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Stability Estimates for Hybrid Coupled Domain Decomposition Methods

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Contents

Introduction

Domain decomposition methods are a well established tool for an efficient numerical solution of partial differential equations, in particular for the coupling of different

- models, i.e., partial differential equations;
- discretization methods such as finite and boundary element methods;
- finite–dimensional trial spaces and their underlying meshes.

Especially when solving boundary value problems in complicated three–dimensional structures, a decomposition of the complex domain into simpler subdomains seems to be advantageous. Then we can replace the global problem by local subproblems, which are linked together by suitable transmission or coupling conditions. The solution of local boundary value problems defines local Dirichlet–Neumann or Neumann–Dirichlet maps. Hence, in domain decomposition methods we need to find the complete Cauchy data on the skeleton. This results in a variational formulation to find either the Dirichlet or Neumann data on the skeleton, and the remaining data are determined by the local problems and the coupling conditions. By solving local Dirichlet boundary value problems we can define local Dirichlet–Neumann maps involving the Steklov–Poincaré operator acting on the given Dirichlet data and some Newton potential to deal with given volume forces. To describe the Steklov–Poincaré operator locally we can use either a variational formulation within the subdomains or we can use boundary integral equations to find different representations of the Steklov–Poincaré operator. Since all of these definitions are given implicitly, we have to define suitable approximations of the local Steklov–Poincaré operators to be used in practical computations. For this we may use either finite or boundary element methods locally leading to a natural algorithm for the coupling of finite and boundary elements. Moreover, since these approximations are defined locally by solving Dirichlet boundary value problems, the underlying meshes do not need to satisfy any compatibility condition.

O. Steinbach: LNM 1809, pp. 1–5, 2003.

The coupling of locally different trial spaces is the main concern of this work. In many situations, for example in case of geometrically singularities or jumping coefficients, one would like to use local trial spaces defined on adaptively refined meshes or of different polynomial degree, or a combination of both. Then the problem arises, how to couple the local trial spaces to get a stable approximation globally.

In [8], a new concept to couple standard finite elements and spectral elements was introduced; this approach finally leads to the Mortar finite element method [9]. Introducing Lagrange multipliers as dual variables, a weak coupling of the primal variables is formulated. Variants of this method are hybrid coupled domain decomposition methods [3]; the hybrid coupling of finite and boundary elements [43]; three–field domain decomposition methods [24]. Since all of these methods can be formulated as saddle point problems, we need to have a certain discrete inf–sup condition to be satisfied [21]. Using a criteria due to Fortin [36], the discrete inf–sup condition is equivalent to the stability of an associated L_2 projection operator in $\widetilde{H}^{1/2}(\Gamma_{ij})$ where Γ_{ij} is the local coupling boundary of the subdomains Ω_i and Ω_j. For globally quasi–uniform meshes, the stability of the L_2 projection operator follows from appropriate error estimates and the use of the inverse inequality. However, for locally quasiuniform meshes such an approach is not applicable. One way out is the use of discrete Sobolev norms [14, 73]. Another possibility is to prove the stability of the L_2 projection operator directly in the scale of Sobolev norms. In [34], the required stability in H^1 was shown for nonuniform triangulations in one and two space dimensions satisfying certain mesh conditions. The analysis is based on decay properties of the L_2 projection and results in conditions which depend on the global behavior of the mesh. In a recent paper [18] we proved the H^1 stability of the L_2 projection onto piecewise linear finite element spaces for arbitrary space dimension. In this case we can formulate explicit local mesh conditions which can be checked easily for a given finite element mesh. This approach can be extended to more general situations, i.e. when using higher order polynomial, dual and biorthogonal basis functions. Biorthogonal basis functions were introduced in [75] to prove the stability of the Mortar finite element method. We will show that biorthogonal basis functions fit in the approach presented here. In the recent literature [7, 13, 49] there is a special emphasis on the numerical analysis of Mortar finite elements in three space dimensions. Using the general approach described in this monograph we are able to design appropriate Lagrange multiplier spaces to be used in hybrid coupled domain decomposition methods. We prove stability estimates for quite general trial spaces assuming only some mild conditions on the underlying mesh. Moreover, by computing some local mesh parameters one is able to control the formulated stability criteria.

Note that the Mortar finite element method is applicable to couple non–matching grids and local trial spaces of different polynomial degree. Another approach to couple locally non-matching grids without the use of Lagrange parameters is based on a domain decomposition formulation with local Dirichlet–Neumann maps. Defining a global trial space on the skeleton, a Galerkin variational problem is formulated for the assembled Steklov–Poincaré operators which correspond to the solution of local Dirichlet boundary value problems. Therefore, an approximation of local Steklov–Poincaré operators is only defined by using local degrees of freedom. When using a boundary element method we need to approximate the conormal derivative, when using a finite element method we need to approximate the solution of a Dirichlet boundary value problem in interior nodes. Since both trial spaces are locally, no compatibility conditions are required. This results in a natural domain decomposition method [68]. For solving a Dirichlet boundary value problem using finite elements we have to extend the given Dirichlet data from the boundary to the domain. In case of nested trial spaces, this can be done by interpolation, otherwise one may use a two–level method globally.

The discretization of several domain decomposition algorithms discussed here leads to linear algebraic systems, where the stiffness matrix is in general positive definite, but either symmetric or block skew–symmetric. Hence, for an efficient iterative solution in parallel, one needs to design special algorithms and almost optimal preconditioners to be used. Note that we will not focus on this topic here, but we refer to [11, 17, 41] for finite element domain decomposition methods; to [14, 38, 70, 74] for multigrid methods for Mortar finite elements; to [27, 51, 60, 69] for boundary element domain decomposition methods; to [16] for positive definite and block skew–symmetric linear systems.

In this work, our main focus is on the formulation and on the stability analysis of hybrid coupled domain decomposition methods. In Chapter 1 we review the definition of Sobolev spaces and give an overview about variational methods for saddle point problems. In particular we discuss a two–fold saddle point formulation. After introducing some notations for finite element spaces we define several L_2 projection operators by Galerkin–Bubnov and Galerkin–Petrov variational problems. To prove the stability of these operators we need to have a bounded projection operator providing local error estimates. For this we recall the definition of quasi interpolation operators from [30].

Based on the equivalence of different stability estimates, and assuming the positive definiteness of a scaled Gram matrix, we prove in Chapter 2 the stability of the L_2 projection in H^s for $s \in (0, 1]$. Then we investigate the required positivity assumption on the scaled Gram matrix by computing its minimal eigenvalue. For piecewise linear finite elements this results in an explicit and easily computable formula. When using higher order polynomial basis functions it is in general impossible to compute the minimal eigenvalue of the scaled Gram matrix in an explicit form. However, for a given mesh we can compute the eigenvalues numerically. We will illustrate the applicability of this approach by using Lagrange polynomials and antiderivatives of Legendre polynomials as local basis functions.

In Chapter 3 we introduce the Dirichlet–Neumann map and define the Steklov–Poincaré operator and the Newton potential by solving related Dirichlet boundary value problems. A first approach is based on a variational formulation using the Dirichlet bilinear form, the second one is based on a symmetric representation of the Steklov–Poincaré operator by using boundary integral operators. Using these representations we obtain results on the mapping properties of the Steklov–Poincaré operator. Since the Steklov–Poincaré is defined via the solution of a Dirichlet boundary value problem, we have to introduce suitable approximations. According to the definitions we use either a finite element method in the domain or a boundary element method on the boundary. Both lead to stable approximations of the Steklov–Poincaré operator. Applying the same ideas we can approximate the Newton potential and therefore we obtain an approximate Dirichlet–Neumann map.

This approximate Dirichlet–Neumann map is used in Chapter 4 for the numerical solution of mixed boundary value problems. The trial space for the unknown Dirichlet data on the boundary is in general independent of the trial space used to approximate the Steklov–Poincaré operator. In a first approach we eliminate the Neumann data while in a second approach we keep the Neumann data as an unknown function in the variational formulation. This is then equivalent to variational formulations using Lagrange multipliers [4, 15]. When using a compatible trial space to approximate the Steklov–Poincaré operator by finite elements, this discrete Steklov–Poincaré operator coincides with the Schur complement of the standard finite element method.

In Chapter 5 we use the same ideas to formulate hybrid coupled domain decomposition methods. Using a global trial space on the skeleton and eliminating the Neumann data by a weak coupling condition across the local interfaces, this gives a variational formulation of the assembled Steklov–Poincaré operators. To approximate the local Steklov–Poincaré operators, we can use trial spaces, which are independent of the trial space on the skeleton. Especially when using finite element approximations of the local Steklov–Poincaré operators, we obtain a method, which includes the coupling of non–matching meshes in a natural way. For a more practical approach we can formulate this method as a two–level algorithm consisting of a global coarse grid space and local fine grid spaces. To be more flexible, one may introduce an additional trial space for the primal variable locally. This leads to a three field domain decomposition method [24] which can be analyzed as a two–fold saddle point problem. To ensure stability, we have to define appropriate trial spaces satisfying the stability conditions as formulated in Chapter 2. When using a strong coupling of the local Neumann data, i.e. using a formulation with Lagrange parameters, we obtain a Mortar finite element method. Using the theory on saddle point problems we can ensure stability and convergence, when the trial spaces are chosen in an appropriate way. To illustrate the applicability of the proposed natural domain decomposition method we describe then a simple numerical experiment. We consider two model problems with jumping coefficients requiring a heterogeneous discretization within the subdomains. Finally, to describe a more practical situation, we consider a three–dimensional problem from linear elastostatics, where the domain is non Lipschitz. A domain decomposition leads to local subproblems where the substructures are Lipschitz domains. The local Steklov–Poincaré operators are then discretized by a symmetric Galerkin boundary element method.

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1

Preliminaries

In this chapter we summarize some results which are needed frequently in the succeeding chapters. In particular, we give a brief introduction to Sobolev spaces, for a detailed presentation, see for example [1, 52, 53]. Following [21] we then describe abstract results for the variational solution of saddle point problems, see also [57]. Then we summarize some basic definitions and properties of general finite element spaces and their underlying triangulations. Using Galerkin–Bubnov and Galerkin–Petrov variational formulations we introduce L_2 projection operators onto finite element spaces. To prove the stability of these operators in a scale of Sobolev spaces we need to have projection operators which are stable and admit local error estimates. Following [30] we introduce quasi–interpolation operators which satisfy both of these requirements.

1.1 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ with $n = 2$ or $n = 3$ be a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$. For $k \in \mathbb{N}_0$ we define the norm

$$||u||_{H^k(\Omega)} := \left\{ \sum_{|\alpha| \le k} ||D^\alpha u||^2_{L_2(\Omega)} \right\}^{1/2} \tag{1.1}$$

while for $0 < s \in \mathbb{R}, s \notin \mathbb{N}$, we define

$$||u||_{H^s(\Omega)} := \left\{ ||u||^2_{H^{[s]}(\Omega)} + |u|^2_{H^s(\Omega)} \right\}^{1/2} \tag{1.2}$$

using the Sobolev–Slobodeckii norm

$$|u|_{H^s(\Omega)} := \left\{ \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2s}} dx dy \right\}^{1/2} . \tag{1.3}$$

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For $s \geq 0$ we introduce the Sobolev spaces

$$\begin{aligned} H^s(\Omega) &:= \overline{C^\infty(\Omega)}^{||\cdot||_{H^s(\Omega)}}, \\ H^s_0(\Omega) &:= \overline{C^\infty_0(\Omega)}^{||\cdot||_{H^s(\Omega)}}, \\ \widetilde{H}^s(\Omega) &:= \overline{C^\infty_0(\Omega)}^{||\cdot||_{H^s(\mathbb{R}^n)}}. \end{aligned}$$

Note that for $s \geq 0$ we have the embedding

$$\widetilde{H}^s(\Omega) \subseteq H^s_0(\Omega). \tag{1.4}$$

In fact, see for example [52, Theorem 3.33],

$$\widetilde{H}^s(\Omega) \;=\; H^s_0(\Omega) \quad \text{provided } s \notin \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \right\}.$$

For $s < 0$ the Sobolev spaces are defined by duality with respect to the $L_2(\Omega)$ inner product,

$$H^s(\Omega) \;:=\; [\widetilde{H}^{-s}(\Omega)]^*, \quad \widetilde{H}^s(\Omega) \;:=\; [H^{-s}(\Omega)]^* \tag{1.5}$$

with norms

$$\begin{aligned} ||f||_{H^s(\Omega)} &:= \sup_{0 \neq v \in \widetilde{H}^{-s}(\Omega)} \frac{|\langle f, v \rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}}, \\ ||f||_{\widetilde{H}^s(\Omega)} &:= \sup_{0 \neq v \in H^{-s}(\Omega)} \frac{|\langle f, v \rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}}. \end{aligned}$$

Let $f \in \widetilde{H}^s(\Omega)$ be given for some $s < 0$. Then,

$$\begin{aligned} ||f||_{H^s(\Omega)} &= \sup_{0 \neq v \in \widetilde{H}^{-s}(\Omega)} \frac{|\langle f, v \rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}} \\ &\leq \sup_{0 \neq v \in H^{-s}(\Omega)} \frac{|\langle f, v \rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}} \;=\; ||f||_{\widetilde{H}^s(\Omega)} \end{aligned} \tag{1.6}$$

and therefore $f \in H^s(\Omega)$. Hence we have the embedding $\widetilde{H}^s(\Omega) \subseteq H^s(\Omega)$ for all $s \in \mathbb{R}$.

In a similar way as above we can define Sobolev spaces on the closed boundary $\Gamma := \partial\Omega$. In particular, we are interested in the case $s \in (0, 1)$ where the norm of the Sobolev space $H^s(\Gamma)$ is given by

$$||u||_{H^s(\Gamma)} \;:=\; \left\{ ||u||^2_{L_2(\Gamma)} + |u|^2_{H^s(\Gamma)} \right\}^{1/2} \tag{1.7}$$

with

$$|u|^2_{H^s(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^{n-1+2s}} ds_x ds_y \,. \tag{1.8}$$

For $s=1$ we use the covariant derivatives to define

$$||u||_{H^1(\Gamma)} := \left\{ ||u||^2_{L_2(\Gamma)} + \sum_{|\alpha|=1} ||D^\alpha u||^2_{L_2(\Gamma)} \right\}. \tag{1.9}$$

Note that for $s>1$ we need stronger assumptions on Ω to define Sobolev spaces $H^s(\Gamma)$, see for example [52]. In particular, if Ω is $C^{k-1,1}$ for $k \geq 0$, then $H^s(\Gamma)$ is well defined for $|s| \leq k$.

Finally, for $s<0$ we define $H^s(\Gamma) := [H^{-s}(\Gamma)]^*$ by duality with respect to the $L_2(\Gamma)$ inner product,

$$||\lambda||_{H^s(\Gamma)} = \sup_{0\neq v\in H^{-s}(\Gamma)} \frac{|\langle \lambda, v\rangle_{L_2(\Gamma)}|}{||v||_{H^{-s}(\Gamma)}}. \tag{1.10}$$

Theorem 1.1. [52, 53] *Let $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$. For any $u \in H^1(\Omega)$ there exists the trace $\gamma_0 u \in H^{1/2}(\Gamma)$ satisfying*

$$||\gamma_0 u||_{H^{1/2}(\Gamma)} \leq c_T \cdot ||u||_{H^1(\Omega)}. \tag{1.11}$$

Vice versa, for any $u \in H^{1/2}(\Gamma)$ there exists a bounded extension $\mathcal{E}u \in H^1(\Omega)$ satisfying $\gamma_0 \mathcal{E} u = u$ and

$$||\mathcal{E}u||_{H^1(\Omega)} \leq c_{IT} \cdot ||u||_{H^{1/2}(\Gamma)}. \tag{1.12}$$

Let $\Gamma_0 \subset \Gamma$ be an open subset of the closed boundary $\Gamma = \partial\Omega$. As in (1.4) we define two kinds of Sobolev spaces on Γ_0,

$$H^s(\Gamma_0) := \left\{ u \,:\, u = U_{|\Gamma_0} \text{ for some } U \in H^s(\Gamma) \right\}$$
$$\widetilde{H}^s(\Gamma_0) := \left\{ u \in H^s(\Gamma) \,:\, \operatorname{supp} u \subseteq \overline{\Gamma}_0 \right\}$$

with norms

$$||u||_{H^s(\Gamma_0)} := \inf_{u=U_{|\Gamma_0}} ||U||_{H^s(\Gamma)}, \quad ||u||_{\widetilde{H}^s(\Gamma_0)} := ||u||_{H^s(\Gamma)}.$$

These two families of spaces are related by duality with respect to $L_2(\Gamma_0)$,

$$[H^s(\Gamma_0)]^* = \widetilde{H}^{-s}(\Gamma_0), \quad [\widetilde{H}^s(\Gamma_0)]^* = H^{-s}(\Gamma_0) \quad \text{for } s \in \mathbb{R}.$$

Note that $\widetilde{H}^s(\Gamma_0)$ is often denoted by $H^s_{00}(\Gamma_0)$.

1.2 Saddle Point Problems

Let X and Π be some Hilbert spaces equipped with norms $||\cdot||_X$ and $||\cdot||_\Pi$, respectively. We assume that there are given some bounded bilinear forms

$$\begin{aligned} a(\cdot,\cdot) &: X \times X \to \mathbb{R}, \\ b(\cdot,\cdot) &: X \times \Pi \to \mathbb{R}. \end{aligned}$$

Note that by the Riesz representation theorem we can identify the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ with bounded operators $A : X \to X^*$ and $B : X \to \Pi^*$, respectively. In particular, for $u \in X$ we define $Au \in X^*$ such that

$$\langle Au, v\rangle \;=\; a(u,v) \quad \text{for all } v \in X$$

and $Bu \in \Pi^*$ satisfying

$$\langle Bu, \mu\rangle \;=\; b(u,\mu) \quad \text{for all } \mu \in \Pi.$$

For given $f \in X^*$ and $g \in \Pi^*$ we consider the saddle point problem to find $(u,\lambda) \in X \times \Pi$ such that

$$\begin{aligned} a(u,v) - b(v,\lambda) &= \langle f, v\rangle \\ b(u,\mu) \qquad\qquad &= \langle g, \mu\rangle \end{aligned} \tag{1.13}$$

for all $(v,\mu) \in X \times \Pi$.

Denote

$$V \;:=\; \ker B \;:=\; \{v \in X \;:\; b(v,\tau) \;=\; 0 \quad \text{for all } \tau \in \Pi\}\,, \tag{1.14}$$

its orthogonal complement

$$V^\perp \;:=\; \{w \in X \;:\; (w,v) = 0 \quad \text{for all } v \in V,\}\,. \tag{1.15}$$

and

$$V^0 \;:=\; \{f \in X^* \;:\; \langle f, v\rangle = 0 \quad \text{for all } v \in V\}\,. \tag{1.16}$$

Theorem 1.2. [12, 21, 57] *Let the bounded bilinear form $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be elliptic on $V = ker B$,*

$$a(v,v) \;\geq\; c_1^A \cdot ||v||_X^2 \quad \textit{for all } v \in V = ker B\,. \tag{1.17}$$

If the bounded bilinear form $b(\cdot,\cdot) : X \times \Pi \to \mathbb{R}$ satisfies the inf–sup condition

$$\inf_{0 \neq \mu \in \Pi} \; \sup_{0 \neq v \in X} \frac{b(v,\mu)}{||v||_X ||\mu||_\Pi} \;\geq\; \gamma_S \;>\; 0\,, \tag{1.18}$$

and if $g \in Im B$, then there exists a unique solution of (1.13) satisfying

$$||u||_X + ||\lambda||_\Pi \;\leq\; c \cdot \{||f||_{X^*} + ||g||_{\Pi^*}\}\,. \tag{1.19}$$

Moreover, $B : V^\perp \to \Pi^*$ *is an isomorphism satisfying*

$$\gamma_S \cdot ||v||_X \;\le\; ||Bv||_{\Pi^*} \quad \text{for all } v \in V^\perp. \tag{1.20}$$

Finally, the operator $B^* : \Pi \to V^0$ *is an isomorphism satisfying*

$$\gamma_S \cdot ||\mu||_\Pi \;\le\; ||B^*\mu||_{X^*} \quad \text{for all } \mu \in \Pi. \tag{1.21}$$

Let $X_h \subset X$ and $\Pi_h \subset \Pi$ be conforming finite dimensional trial spaces. The Galerkin variational formulation of the saddle point problem (1.13) is to find $(u_h, \lambda_h) \in X_h \times \Pi_h$ such that

$$\begin{aligned} a(u_h, v_h) - b(v_h, \lambda_h) &= \langle f, v_h\rangle \\ b(u_h, \mu_h) \qquad\qquad &= \langle g, \mu_h\rangle \end{aligned} \tag{1.22}$$

for all $(v_h, \mu_h) \in X_h \times \Pi_h$. As in the continuous case we define

$$V_h \;:=\; \{v_h \in X_h \;:\; b(v_h, \tau_h) \;=\; 0 \quad \text{for all } \tau_h \in \Pi_h\} \tag{1.23}$$

and assume that the bilinear form $a(\cdot, \cdot)$ is V_h–elliptic,

$$a(v_h, v_h) \;\ge\; \widetilde{c}_1^A \cdot ||v_h||_X^2 \quad \text{for all } v_h \in V_h. \tag{1.24}$$

Theorem 1.3. [12, 21, 57] *Let the assumptions of Theorem 1.2 be satisfied and let the bilinear form* $a(\cdot, \cdot)$ *be* V_h*–elliptic. Let* $V_h \subset V$*. If the discrete inf–sup condition*

$$\inf_{0 \ne \mu_h \in \Pi_h} \; \sup_{0 \ne v_h \in X_h} \frac{b(v_h, \mu_h)}{||v_h||_X ||\mu_h||_\Pi} \;\ge\; \widetilde{\gamma}_S \;>\; 0 \tag{1.25}$$

is valid, then there exists a unique solution of (1.22) satisfying the error estimates

$$||u - u_h||_X \le c_1 \cdot \inf_{v_h \in X_h} ||u - v_h||_X,$$

$$||\lambda - \lambda_h||_\Pi \le c_2 \cdot \left\{ \inf_{v_h \in X_h} ||u - v_h||_X + \inf_{\mu_h \in \Pi_h} ||\lambda - \mu_h||_\Pi \right\}.$$

Hence we have convergence when assuming some approximation properties of X_h and Π_h. The crucial assumption of the preceding theorem is the discrete inf–sup condition (1.25). To characterize this condition we use a criteria due to Fortin [36]:

Theorem 1.4. *Assume that the continuous inf–sup condition (1.18) is satisfied. Let* $P_h : X \to X_h$ *be a projection operator satisfying the orthogonality*

$$b(v - P_h v, \mu_h) \;=\; 0 \qquad \text{for all } \mu_h \in \Pi_h$$

and the stability estimate

$$||P_h v||_X \;\le\; c_S \cdot ||v||_X \qquad \text{for all } v \in X.$$

Then, (1.25) holds with $\widetilde{\gamma}_S = \gamma_S / c_S$*.*

Instead of the saddle point problem (1.13) we now consider a two–fold saddle point problem. Let X, Π_1 and Π_2 be some Hilbert spaces with norms $||\cdot||_X$, $||\cdot||_{\Pi_1}$ and $||\cdot||_{\Pi_2}$. Then we want to find $(u,\lambda,w) \in X \times \Pi_1 \times \Pi_2$ such that

$$\begin{aligned} b_1(u,\mu) - b_2(\mu,w) &= \langle f_1,\mu\rangle \\ -b_1(v,\lambda) + a(u,v) &= \langle f_2,v\rangle \\ b_2(\lambda,z) &= \langle g,z\rangle \end{aligned} \tag{1.26}$$

for all $(v,\mu,z) \in X \times \Pi_1 \times \Pi_2$. Here,

$$\begin{aligned} a(\cdot,\cdot) &: X \times X \to \mathbb{R}, \\ b_1(\cdot,\cdot) &: X \times \Pi_1 \to \mathbb{R}, \\ b_2(\cdot,\cdot) &: \Pi_1 \times \Pi_2 \to \mathbb{R} \end{aligned}$$

are bounded bilinear forms implying, by the Riesz representation theorem, bounded operators $A : X \to X^*$, $B_1 : X \to \Pi_1^*$ and $B_2 : \Pi_1 \to \Pi_2^*$, respectively.

Two–fold saddle point problems (1.26) appear in many applications: the coupling of mixed finite elements and symmetric boundary element methods [37]; hybrid boundary element methods [64] or three–field domain decomposition methods, see [24].

To prove unique solvability of the two–fold saddle point problem (1.26) we apply Theorem 1.2 twice. As in (1.14) we define

$$\ker B_2 := \{\mu \in \Pi_1 \; : \; b_2(\mu,z) = 0 \quad \text{for all } z \in \Pi_2\}\,. \tag{1.27}$$

Moreover,

$$\ker_{B_2} B_1 := \{v \in X \; : \; b_1(v,\mu) = 0 \quad \text{for all } \mu \in \ker B_2\}\,. \tag{1.28}$$

Theorem 1.5. *Let the bounded bilinear form $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be elliptic on $ker_{B_2} B_1$,*

$$a(v,v) \ \geq\ c \cdot ||v||_X^2 \quad \textit{for all } v \in ker_{B_2} B_1\,. \tag{1.29}$$

Let the bounded bilinear forms $b_1(\cdot,\cdot) : X \times \Pi_1$ and $b_2(\cdot,\cdot) : \Pi_1 \times \Pi_2$ satisfy the inf–sup conditions

$$\inf_{0\neq\mu\in\Pi_1} \sup_{0\neq v\in X} \frac{b_1(v,\mu)}{||v||_X ||\mu||_{\Pi_1}} \geq \gamma_S \ >\ 0, \tag{1.30}$$

$$\inf_{0\neq z\in\Pi_2} \sup_{0\neq\mu\in\Pi_1} \frac{b_2(\mu,z)}{||\mu||_{\Pi_1} ||z||_{\Pi_2}} \geq \gamma_S \ >\ 0. \tag{1.31}$$

If $g \in Im\, B_2$ and $f_1 \in Im\, B_1$ is satisfied, then there exists a unique solution $(u,\lambda,w) \in X \times \Pi_1 \times \Pi_2$ of the two–fold saddle point problem (1.26) satisfying the stability estimate

$$||u||_X + ||\lambda||_{\Pi_1} + ||w||_{\Pi_2} \ \leq\ c \cdot \left\{ ||f_1||_{\Pi_1^*} + ||f_2||_{X^*} + ||g||_{\Pi_2^*} \right\}. \tag{1.32}$$

Proof. Since we assume $g \in \mathrm{Im} B_2$ there exists a $\lambda^\perp \in (\ker B_2)^\perp$ such that

$$b_2(\lambda^\perp, z) \;=\; \langle B_2\lambda^\perp, z\rangle \;=\; \langle g, z\rangle \quad \text{for all } z \in \Pi_2.$$

Moreover,

$$||\lambda^\perp||_{\Pi_1} \;\le\; c\cdot ||g||_{\Pi_2^*}.$$

Hence we have to find $(u, \lambda_0) \in X \times \ker B_2$ such that

$$\begin{aligned} b_1(u,\mu) &= \langle f_1, \mu\rangle \\ -b_1(v,\lambda_0) + a(u,v) \;&= \langle f_2, v\rangle + b_1(v,\lambda^\perp) \end{aligned}$$

for all $(v, \mu) \in X \times \ker B_2$. Note that $b_2(w,\mu) = \langle B_2\mu, w\rangle = 0$ for $\mu \in \ker B_2$. Using $f_1 \in \mathrm{Im} B_1$ there exists a $u^\perp \in (\ker_{B_2} B_1)^\perp$ such that

$$b_1(u^\perp, \mu) \;=\; \langle f_1, \mu\rangle \quad \text{for all } \mu \in \ker B_2\,.$$

Again we have the stability estimate

$$||u^\perp||_X \;\le\; c\cdot ||f_1||_{\Pi_1^*}.$$

It remains to find $u_0 \in \ker_{B_2} B_1$ such that

$$a(u_0, v) \;=\; \langle f_2, v\rangle + b_1(v,\lambda^\perp) \quad \text{for all } v \in \ker_{B_2} B_1.$$

Note that $b_1(v,\lambda_0) \;=\; \langle B_1 v, \lambda_0\rangle = 0$ for $v \in \ker_{B_2} B_1$. Since the bilinear form $a(\cdot,\cdot)$ is assumed to be elliptic for $v \in \ker_{B_2} B_1$, there exists a unique solution $u_0 \in \ker_{B_2} B_1$ satisfying

$$||u_0||_X \;\le\; c\cdot \left\{||f_2||_{X^*} + ||\lambda^\perp||_{\Pi_1}\right\}.$$

Therefore, $u := u_0 + u^\perp$ is the unique solution of (1.26) satisfying

$$||u||_X \;\le\; c\cdot \left\{||f_1||_{\Pi_1^*} + ||f_2||_{X^*} + ||g||_{\Pi_2^*}\right\}.$$

Now we can find $\lambda_0 \in \ker B_2$ such that

$$b_1(v,\lambda_0) \;=\; \langle B_1'\lambda_0, v\rangle \;=\; \langle Au - f_2 - B_1'\lambda^\perp, v\rangle \quad \text{for all } v \in X\,.$$

Since $Au - f_2 - B_1'\lambda^\top \in (\ker_{B_2} B_1)^0$ is satisfied, we can apply Theorem 1.2 to get a unique solution $\lambda_0 \in \ker B_2$. Hence, $\lambda := \lambda_0 + \lambda^\perp \in \Pi_1$ is the unique solution of (1.26) satisfying

$$||\lambda||_{\Pi_1} \;\le\; c\cdot \left\{||f_1||_{\Pi_1^*} + ||f_2||_{X^*} + ||g||_{\Pi_2^*}\right\}.$$

Finally we can find $w \in \Pi_2$ such that

$$b_2(w,\mu) \;=\; \langle B_2' w, \mu\rangle \;=\; \langle B_1 u - f_1, \mu\rangle \quad \text{for all } \mu \in \Pi_1\,.$$

Since $B_1 u - f_1 \in (\ker B_2)^0$ we can apply Theorem 1.2 to get $w \in \Pi_2$ solving the saddle point problem (1.26) and satisfying the stability estimate

$$||w||_{\Pi_2} \;\le\; c\cdot \left\{||f_1||_{\Pi_1^*} + ||f_2||_{X^*} + ||g||_{\Pi_2^*}\right\}. \quad \square$$

Let $X_h \subset X$, $\Pi_{1,h} \subset \Pi_1$ and $\Pi_{2,h} \subset \Pi_2$ be conforming finite dimensional trial spaces. The Galerkin variational formulation of the two–fold saddle point problem is: find $(u_h, \lambda_h, w_h) \in X_h \times \Pi_{1,h} \times \Pi_{h,2}$ such that

$$\begin{aligned} & b_1(u_h,\mu_h) - b_2(\mu_h,w_h) &&= \langle f_1,\mu_h\rangle \\ -&b_1(v_h,\lambda_h) + a(u_h,v_h) &&= \langle f_2, v_h\rangle \\ & b_2(\lambda_h,z_h) &&= \langle g, z_h\rangle \end{aligned} \tag{1.33}$$

for all $(v_h,\mu_h,z_h) \in X_h \times \Pi_{1,h} \times \Pi_{2,h}$. We assume that the bilinear form $a(\cdot,\cdot)$ is also elliptic on

$$\ker_{B_{2,h}} B_{1,h} := \{v_h \in X_h \ : \ b_1(v_h,\mu_h) = 0 \quad \text{for all } \mu_h \in \ker B_{2,h}\}$$

where

$$\ker B_{2,h} := \{\mu_h \in \Pi_{1,h} \ : \ b_2(\mu_h, z_h) = 0 \quad \text{for all } z_h \in \Pi_{2,h}\}\,.$$

As in Theorem 1.3 we can prove unique solvability of the two–fold Galerkin saddle point problem (1.33) when assuming some discrete inf–sup conditions:

Theorem 1.6. *Let the assumptions of Theorem 1.5 be satisfied and let the bilinear form $a(\cdot,\cdot)$ be elliptic on $\ker_{B_{2,h}} B_{1,h}$. If in addition the discrete inf–sup conditions*

$$\inf_{0\neq\mu_h\in\Pi_{h,1}} \sup_{0\neq v_h\in X_h} \frac{b_1(v_h,\mu_h)}{||v_h||_X ||\mu_h||_{\Pi_1}} \geq \widetilde{\gamma}_S > 0, \tag{1.34}$$

$$\inf_{0\neq z_h\in\Pi_{h,2}} \sup_{0\neq \mu_h\in\Pi_{1,h}} \frac{b_2(\mu_h,z_h)}{||\mu_h||_{\Pi_1} ||z_h||_{\Pi_2}} \geq \widetilde{\gamma}_S > 0 \tag{1.35}$$

are valid, then there exists a unique solution of (1.22) satisfying the error estimates

$$\begin{aligned} &||u-u_h||_X + ||\lambda-\lambda_h||_{\Pi_1} + ||w-w_h||_{\Pi_2} \\ &\qquad \leq c\cdot\left\{\inf_{v_h\in X_h}||u-v_h||_X + \inf_{\mu_h\in\Pi_{h,1}}||\lambda-\mu_h||_{\Pi_1} + \inf_{z_h\in\Pi_{h,2}}||w-z_h||_{\Pi_2}\right\}. \end{aligned}$$

1.3 Finite Element Spaces

Let $\mathcal{T}$ be a m–dimensional bounded and sufficiently smooth, open or closed manifold. For simplicity in the presentation we assume that for $m = 1,2$ $\mathcal{T} \subseteq \partial\Omega$ where $\Omega \subset \mathbb{R}^{m+1}$ is a bounded polygonal or polyhedral domain. For $m = 3$ let $\mathcal{T} = \Omega \subset \mathbb{R}^m$ be a polyhedron. Let $\mathcal{T}_N = \{\tau_\ell\}_{\ell=1}^N$ be a family of admissible finite element meshes consisting of N finite elements τ_ℓ which are assumed to be open. We assume that a finite element τ_ℓ is either a line segment ($m = 1$), a triangle ($m = 2$) or a tetrahedron ($m = 3$). Note that for more general $\mathcal{T}$ one may consider either a sequence of polygonal/polyhedral approximations $\mathcal{T}_N$ or higher order finite elements τ_ℓ. The first approach will require an appropriate error analysis based on the Strang lemma [29] while for the second approach we can use the same techniques as described in this monograph. However, the computations will become more technically. Hence, we will restrict ourself to consider the simple case of a polygonal or polyhedral bounded domain only.

For each finite element τ_ℓ we define as local mesh parameters: the volume Δ_ℓ, the associated local mesh size h_ℓ and the diameter d_ℓ,

$$\Delta_\ell := \int_{\tau_\ell} dx, \quad h_\ell := \Delta_\ell^{1/m}, \quad d_\ell := \sup_{x,y\in\tau_\ell} |x-y| \quad \text{for } \ell = 1,\ldots,N. \tag{1.36}$$

We assume that all finite elements τ_ℓ are shape regular, in particular, there exists a positive constant c_R independent of N such that

$$1 \geq \frac{h_\ell}{d_\ell} \geq c_R > 0\,. \tag{1.37}$$

Let

$$h_{\max} := \max_{\ell=1,\ldots,N} h_\ell, \qquad h_{\min} := \min_{\ell=1,\ldots,N} h_\ell. \tag{1.38}$$

The family of finite element meshes $\mathcal{T}_N$ is said to be *globally quasi-uniform* if there is a constant $c_G \geq 1$ independent of N such that

$$h_{\max}/h_{\min} \leq c_G\,. \tag{1.39}$$

As usual we put $h := h_{\max}$ to be the global mesh parameter associated with the mesh $\mathcal{T}_N$. In this work we are mainly interested in *locally quasi-uniform* finite element meshes $\mathcal{T}_N$ where the ratio

$$h_\ell/h_j \leq c_L \tag{1.40}$$

is bounded by a uniform constant c_L whenever the elements τ_ℓ and τ_j are neighbored, in particular, they share either a common vertex, edge or face.

Let

$$V_h := \operatorname{span}\{\varphi_k\}_{k=1}^M \subset H^1(\mathcal{T}) \tag{1.41}$$

be an associated family of finite dimensional trial spaces spanned by piecewise polynomial basis functions with local support. We assume that there holds some approximation property in V_h, which will be specified later. Furthermore, we assume that an inverse inequality is valid locally,

$$||v_h||_{H^s(\tau_\ell)} \leq c_I \cdot h_\ell^{-s} \cdot ||v_h||_{L_2(\tau_\ell)} \quad \text{for all } v_h \in V_h,\ s \in (0,1]. \tag{1.42}$$

Hence, for $s = 1$ we have an inverse inequality globally,

$$||v_h||^2_{H^1(\mathcal{T})} \;\le\; c_I \cdot \sum_{\ell=1}^{N} h_\ell^{-2} \cdot ||v_h||^2_{L_2(\tau_\ell)} \quad \text{for } v_h \in V_h. \tag{1.43}$$

Now, if the mesh is globally quasi-uniform, there holds also a global inverse inequality for $s \in (0,1]$,

$$||v_h||_{H^s(\mathcal{T})} \;\le\; c \cdot h^{-s} \cdot ||v_h||_{L_2(\mathcal{T})} \quad \text{for all } v_h \in V_h. \tag{1.44}$$

For an arbitrary finite element τ_ℓ the local trial space $V_h(\tau_\ell) := V_{h|\tau_\ell}$ is spanned by local shape functions with $M_\ell := \dim V_h(\tau_\ell)$. In particular,

$$V_h(\tau_\ell) \;:=\; \left\{\varphi_i^\ell \;:\; \exists \varphi_k \in V_h \;:\; \varphi_i^\ell(x) \;=\; \varphi_k(x) \quad \text{for } x \in \tau_\ell\right\}. \tag{1.45}$$

For each local index $i = 1, \ldots, M_\ell$ we define $k_i \in \{1, \ldots, M\}$ as associated global index such that $\varphi_{k_i|\tau_\ell} = \varphi_i^\ell$. For $\ell = 1, \ldots, N$ we introduce local Gram matrices as

$$G_\ell[j,i] \;=\; \langle \varphi_i^\ell, \varphi_j^\ell \rangle_{L_2(\tau_\ell)} \quad \text{for } i,j = 1, \ldots, M_\ell \tag{1.46}$$

and the diagonal matrices

$$D_\ell \;=\; \operatorname{diag} G_\ell \;=\; \operatorname{diag} \left(||\varphi_i^\ell||^2_{L_2(\tau_\ell)} \right)_{i=1}^{M_\ell}. \tag{1.47}$$

Assumption 1.1 *We assume that*

$$c_1^G \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \;\le\; (G_\ell \underline{x}_\ell, \underline{x}_\ell) \;\le\; c_2^G \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \tag{1.48}$$

holds uniformly for all $\underline{x}_\ell \in \mathbb{R}^{M_\ell}$ *(*$\ell = 1, \ldots, N$*) with positive constants* c_1^G, c_2^G *independent of* N*.*

As it will be shown in Chapter 2, Assumption 1.1 holds for a wide class of trial spaces, in particular for piecewise polynomials. In Section 2.1 we consider standard piecewise linear finite elements, while in Section 2.2 we deal with a piecewise constant trial space defined on a dual mesh. Finally, in Section 2.3 we investigate finite element spaces of locally higher order polynomial degree. However, the constants c_1^G and c_2^G may then depend on the polynomial degree used.

For each basis function $\varphi_k \in V_h$, $k = 1, \ldots, M$, we denote its support by

$$\omega_k \;:=\; \operatorname{supp}\{\varphi_k\} \tag{1.49}$$

and we define related trial spaces locally by

$$V_h(\omega_k) \;:=\; \left\{\varphi_{j|\omega_k} \;:\; \varphi_j \in V_h\right\}. \tag{1.50}$$

To describe the relations between the basis functions φ_k for $k = 1, \ldots, M$ and the finite elements τ_ℓ for $\ell = 1, \ldots, N$ we define two index sets as follows:

$$I(k) := \{\ell \in \{1,\ldots,N\} \ : \ \tau_\ell \cap \omega_k \neq \emptyset\} \quad \text{for } k = 1,\ldots,M, \tag{1.51}$$

$$J(\ell) := \{k \in \{1,\ldots,M\} \ : \ \omega_k \cap \tau_\ell \neq \emptyset\} \quad \text{for } \ell = 1,\ldots,N. \tag{1.52}$$

In particular, $I(k)$ defines all finite elements τ_ℓ where the basis function φ_k is not identically zero. Vice versa, $J(\ell)$ describes all basis functions φ_k which are partially nonzero inside τ_ℓ. Since the family of meshes is assumed to be locally quasi-uniform, for each basis function $\varphi_k \in V_h$ we can define an associated mesh size $\hat{h}_k$ satisfying

$$\frac{1}{c_Q} \le \frac{\hat{h}_k}{h_\ell} \le c_Q \qquad \text{for all } \ell \in I(k),\ k = 1,\ldots,M \tag{1.53}$$

with a global constant $c_Q \ge 1$. For example, we can define $\hat{h}_k$ as follows:

i. Averaging of local mesh sizes [18]:

$$\hat{h}_k := \frac{1}{\# I(k)} \sum_{\ell \in I(k)} h_\ell \qquad \text{for } k = 1,\ldots,M. \tag{1.54}$$

ii. Mass of basis functions:

$$\hat{h}_k := ||\varphi_k||^{2/m}_{L_2(\omega_k)} \quad \text{for } k = 1,\ldots,M. \tag{1.55}$$

iii. In case of nodal basis functions we can use the minimal distance of associated vertices, see [26]:

$$\hat{h}_k := \min_{x_j \in \tau_\ell, \ell \in I(k), j \neq k} |x_k - x_j| \quad \text{for } k = 1,\ldots,M. \tag{1.56}$$

iv. In case of a globally quasi-uniform mesh we can choose

$$\hat{h}_k := h \quad \text{for all } k = 1,\ldots,M \tag{1.57}$$

using the global mesh size h.

As it will be seen later, the actual choice of $\hat{h}_k$ does not influence the validity of (1.53). However, for basis functions of higher order polynomial degree, the choice (1.55) seems to be favorable.

Let

$$W_h := \text{span}\{\psi_k\}_{k=1}^M \subset L_2(\mathcal{T}) \tag{1.58}$$

be an appropriately defined *dual finite element space* with

$$\dim W_h = \dim V_h = M$$

where the underlying mesh is given by $\widetilde{\mathcal{T}}_{\widetilde{N}} = \{\widetilde{\tau}_\ell\}_{\ell=1}^{\widetilde{N}}$ consisting of $\widetilde{N}$ finite elements. In Section 2.2 we will describe an example for a construction of W_h. In what follows we need to formulate a few assumptions on W_h to be satisfied.

By $W_h(\tau_\ell)$ we denote the local trial spaces defined by the restriction of W_h onto τ_ℓ:

$$W_h(\tau_\ell) := \left\{\psi_i^\ell : \exists\psi_k \in W_h : \psi_i^\ell(x) = \psi_k(x) \quad \text{for } x \in \tau_\ell\right\}. \tag{1.59}$$

We assume that

$$\dim W_h(\tau_\ell) = \dim V_h(\tau_\ell) = M_\ell \,.$$

As in (1.46) we define the local Gram matrix

$$\hat{G}_\ell[j,i] = \langle \psi_i^\ell, \psi_j^\ell\rangle_{L_2(\tau_\ell)} \quad \text{for } i,j = 1,\ldots,M_\ell. \tag{1.60}$$

In addition, we also define generalized local Gram matrices $\widetilde{G}_\ell$ by

$$\widetilde{G}_\ell[j,i] = \langle \varphi_i^\ell, \psi_j^\ell\rangle_{L_2(\tau_\ell)} \quad \text{for } i,j = 1,\ldots,M_\ell. \tag{1.61}$$

Assumption 1.2 *We assume that*

$$c_1^{\widetilde{G}} \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \le (\widetilde{G}_\ell \underline{x}_\ell, \underline{x}_\ell) \le c_2^{\widetilde{G}} \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \tag{1.62}$$

and

$$c_1^{\hat{G}} \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \le (\hat{G}_\ell \underline{x}_\ell, \underline{x}_\ell) \le c_2^{\hat{G}} \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \tag{1.63}$$

hold uniformly for all $\underline{x}_\ell \in \mathbb{R}^{M_\ell}$ *(*$\ell = 1,\ldots,N$*) with positive constants* $c_1^{\tilde{G}}$, $c_2^{\tilde{G}}$ *and* $c_1^{\hat{G}}$, $c_2^{\hat{G}}$*, respectively.*

Note that in the case $V_h = W_h$ we have $G_\ell = \widetilde{G}_\ell = \hat{G}_\ell$ and Assumption 1.2 coincides with Assumption 1.1. We finally assume that the basis functions of both finite element spaces V_h and W_h are normalized, in particular,

$$||\varphi_k||^2_{L_2(\mathcal{T})} \simeq \hat{h}_k^m, \qquad ||\psi_k||^2_{L_2(\mathcal{T})} \simeq \hat{h}_k^m, \quad k = 1,\ldots,M.$$

1.4 Projection Operators

The standard Galerkin L_2 projection

$$Q_h : L_2(\mathcal{T}) \to V_h \subset H^1(\mathcal{T})$$

of a given function $u \in L_2(\mathcal{T})$ is defined by a Galerkin–Bubnov variational problem: find $Q_h u \in V_h$ such that

$$\langle Q_h u, v_h\rangle_{L_2(\mathcal{T})} = \langle u, v_h\rangle_{L_2(\mathcal{T})} \quad \text{for all } v_h \in V_h\,. \tag{1.64}$$

This is equivalent to a system of linear equations, $G_h \underline{u} = \underline{f}$, where the stiffness matrix is given by

$$G_h[j,i] = \langle \varphi_i, \varphi_j\rangle_{L_2(\mathcal{T})} \quad \text{for all } i,j = 1,\ldots,M,$$

and the right hand side is defined by

$$f_j = \langle u, \varphi_j \rangle_{L_2(\mathcal{T})} \quad \text{for } j = 1, \ldots, M.$$

Let

$$D_h := \operatorname{diag} G_h = \operatorname{diag} \left(||\varphi_k||^2_{L_2(\mathcal{T})} \right)_{k=1}^{M} .$$

As a consequence of Assumption 1.1 there hold the spectral equivalence inequalities

$$c_1^G \cdot (D_h \underline{u}, \underline{u}) \leq (G_h \underline{u}, \underline{u}) \leq c_2^G \cdot (D_h \underline{u}, \underline{u}) \quad \text{for all } \underline{u} \in \mathbb{R}^M \tag{1.65}$$

with the same constants as in (1.48). Hence we have unique solvability of (1.64). Due to

$$||Q_h u||^2_{L_2(\mathcal{T})} = \langle Q_h u, Q_h u \rangle_{L_2(\mathcal{T})} = \langle u, Q_h u \rangle_{L_2(\mathcal{T})} \leq ||u||_{L_2(\mathcal{T})} ||Q_h u||_{L_2(\mathcal{T})}$$

there holds the trivial stability estimate

$$||Q_h u||_{L_2(\mathcal{T})} \leq ||u||_{L_2(\mathcal{T})} \quad \text{for all } u \in L_2(\mathcal{T}). \tag{1.66}$$

Moreover, by Cea's lemma,

$$||u - Q_h u||_{L_2(\mathcal{T})} \leq \inf_{v_h \in V_h} ||u - v_h||_{L_2(\mathcal{T})}$$

and we have convergence when assuming an approximation property of V_h.

Instead of a Galerkin–Bubnov approach we now use a Galerkin–Petrov scheme to define a generalized L_2 projection

$$\widetilde{Q}_h : L_2(\mathcal{T}) \to V_h \subset H^1(\mathcal{T}).$$

For a given $u \in L_2(\mathcal{T})$ this reads: find $\widetilde{Q}_h u \in V_h$ such that

$$\langle \widetilde{Q}_h u, w_h \rangle_{L_2(\mathcal{T})} = \langle u, w_h \rangle_{L_2(\mathcal{T})} \quad \text{for all } w_h \in W_h. \tag{1.67}$$

This is equivalent to a system of linear equations, $\widetilde{G}_h \underline{u} = \underline{f}$, where the stiffness matrix is given by

$$\widetilde{G}_h[j,i] = \langle \varphi_j, \psi_j \rangle_{L_2(\mathcal{T})} \quad \text{for all } i, j = 1, \ldots, M. \tag{1.68}$$

The unique solvability of the variational problem (1.67) is based on Assumption 1.2:

Lemma 1.7. *Let Assumptions 1.1 and 1.2 be satisfied. Then,*

$$c_{S,0} \cdot ||v_h||_{L_2(\mathcal{T})} \leq \sup_{0 \neq w_h \in W_h} \frac{|\langle v_h, w_h \rangle_{L_2(\mathcal{T})}|}{||w_h||_{L_2(\mathcal{T})}} \tag{1.69}$$

holds for all $v_h \in V_h$ with a positive constant $c_{S,0}$.

Proof. Let

$$v_h(x) \,=\, \sum_{k=1}^{M} v_k \varphi_k(x) \in V_h$$

be arbitrary but fixed. We define

$$w_h^*(x) \,=\, \sum_{k=1}^{M} v_k \psi_k(x) \in W_h \,.$$

Due to Assumption 1.2 we have

$$\begin{aligned}\langle v_h, w_h^* \rangle_{L_2(\mathcal{T})} = \sum_{\ell=1}^{N} \langle v_h, w_h^* \rangle_{L_2(\tau_\ell)} \,&=\, \sum_{\ell=1}^{N} (\widetilde{G}_\ell \underline{v}_\ell, \underline{v}_\ell) \\ \geq c_1^{\widetilde{G}} \cdot \sum_{\ell=1}^{N} (D_\ell \underline{v}_\ell, \underline{v}_\ell) \,&=\, c_1^{\widetilde{G}} \cdot (D_h \underline{v}, \underline{v})\end{aligned}$$

as well as

$$\begin{aligned}||w_h^*||^2_{L_2(\mathcal{T})} = \sum_{\ell=1}^{N} \langle w_h^*, w_h^* \rangle_{L_2(\tau_\ell)} \,&=\, \sum_{\ell=1}^{N} (\hat{G}_\ell \underline{v}_\ell, \underline{v}_\ell) \\ \leq c_1^{\hat{G}} \cdot \sum_{\ell=1}^{N} (D_\ell \underline{v}_\ell, \underline{v}_\ell) \,&=\, c_2^{\hat{G}} \cdot (D_h \underline{v}, \underline{v}) \,.\end{aligned}$$

Hence, using Assumption 1.1 and (1.65),

$$\begin{aligned}\frac{|\langle v_h, w_h^* \rangle_{L_2(\mathcal{T})}|}{||w_h^*||_{L_2(\mathcal{T})}} &\geq \frac{c_1^{\widetilde{G}}}{\sqrt{c_2^{\hat{G}}}} \cdot (D_h \underline{v}, \underline{v})^{1/2} \\ &\geq \frac{c_1^{\widetilde{G}}}{\sqrt{c_2^{\hat{G}} c_2^{G}}} \cdot (G_h \underline{v}, \underline{v})^{1/2} \,=\, c_{S,0} \cdot ||v_h||_{L_2(\mathcal{T})}. \quad \square\end{aligned}$$

Applying standard arguments, we have unique solvability of (1.67), the stability estimate

$$||\widetilde{Q}_h u||_{L_2(\mathcal{T})} \,\leq\, \frac{1}{c_{S,0}} \cdot ||u||_{L_2(\mathcal{T})} \quad \text{for all } u \in L_2(\mathcal{T}) \tag{1.70}$$

as well as the quasi optimal error estimate

$$||u - \widetilde{Q}_h u||_{L_2(\mathcal{T})} \,\leq\, (1 + \frac{1}{c_{S,0}}) \cdot \inf_{v_h \in V_h} ||u - v_h||_{L_2(\mathcal{T})} \tag{1.71}$$

yielding convergence from an approximation property of V_h.

As in (1.67) we can define an adjoint projection operator

$$\widetilde{Q}_h^* : L_2(\mathcal{T}) \to W_h \subset L_2(\mathcal{T})$$

by the Galerkin–Petrov variational problem: find $\widetilde{Q}_h^* w \in W_h$ such that

$$\langle \widetilde{Q}_h^* w, v_h \rangle_{L_2(\mathcal{T})} = \langle w, v_h \rangle_{L_2(\mathcal{T})} \quad \text{for all } v_h \in V_h. \tag{1.72}$$

The stability and error analysis of (1.72) is again based on Assumption 1.2 and can be done as described above. In particular,

$$||\widetilde{Q}_h^* w||_{L_2(\mathcal{T})} \le \frac{1}{c_{S,0}} \cdot ||w||_{L_2(\mathcal{T})} \quad \text{for all } w \in L_2(\mathcal{T}). \tag{1.73}$$

Note that, by definition,

$$\langle \widetilde{Q}_h v, w \rangle_{L_2(\mathcal{T})} = \langle v, \widetilde{Q}_h^* w \rangle_{L_2(\mathcal{T})} \quad \text{for all } v, w \in L_2(\mathcal{T}). \tag{1.74}$$

Moreover, in the case $V_h = W_h$ the projection operators Q_h, $\widetilde{Q}_h$ and $\widetilde{Q}_h^*$ coincide.

To this end, we introduce an additional operator

$$\Pi_h^s : H^s(\mathcal{T}) \to W_h \subset \widetilde{H}^{-s}(\mathcal{T})$$

for some $s \in (0,1]$ defined by the variational formulation

$$\langle \Pi_h^s u, v_h \rangle_{L_2(\mathcal{T})} = \langle u, v_h \rangle_{H^s(\mathcal{T})} \quad \text{for all } v_h \in V_h \subset H^s(\mathcal{T}). \tag{1.75}$$

Note that the stiffness matrix $\widetilde{G}_h^\top$ of this variational problem is the same as for the projection operator $\widetilde{Q}_h^*$. Hence we have unique solvability when assuming Assumption 1.2.

In the next chapter we will prove that the operators

$$\begin{aligned}
\widetilde{Q}_h &: H^s(\mathcal{T}) \to V_h \subset H^s(\mathcal{T}), \\
\widetilde{Q}_h^* &: \widetilde{H}^{-s}(\mathcal{T}) \to W_h \subset \widetilde{H}^{-s}(\mathcal{T}), \\
\Pi_h^s &: H^s(\mathcal{T}) \to W_h \subset \widetilde{H}^{-s}(\mathcal{T})
\end{aligned}$$

are bounded for $s \in [0,1]$ when some appropriate assumptions are satisfied.

For a globally quasi-uniform mesh $\mathcal{T}_N$ we have the following result:

Theorem 1.8. *Let the family of meshes $\mathcal{T}_N$ be globally quasi-uniform, in particular we assume the global inverse inequality (1.44). Then, for $s \in [0,1]$,*

$$||\widetilde{Q}_h||_{H^s(\mathcal{T})} \le c_S \cdot ||u||_{H^s(\mathcal{T})} \quad \textit{for all } u \in H^s(\mathcal{T}). \tag{1.76}$$

Proof. For $s = 0$ (1.76) is (1.70). For $s \in (0, 1]$ let $Q_h^s : H^s(\mathcal{T}) \to V_h \subset H^s(\mathcal{T})$ be defined by

$$\langle Q_h^s u, v_h \rangle_{H^s(\mathcal{T})} = \langle u, v_h \rangle_{H^s(\mathcal{T})} \quad \text{for all } v_h \in V_h.$$

Obviously,

$$||Q_h^s u||_{H^s(\mathcal{T})} \leq ||u||_{H^s(\mathcal{T})} \quad \text{for all } u \in H^s(\mathcal{T}).$$

Moreover, applying the Aubin–Nitsche trick, we have the error estimate

$$||u - Q_h^s u||_{L_2(\Omega)} \leq c \cdot h^s \cdot |u|_{H^s(\mathcal{T})}.$$

Using the triangle inequality, the global inverse inequality and the L_2 stability of $\widetilde{Q}_h$,

$$\begin{aligned} ||\widetilde{Q}_h u||_{H^s(\mathcal{T})} &\leq ||Q_h^s u||_{H^s(\mathcal{T})} + ||\widetilde{Q}_h u - Q_h^s u||_{H^s(\mathcal{T})} \\ &\leq ||u||_{H^s(\mathcal{T})} + c \cdot h^{-s} \cdot ||\widetilde{Q}_h u - Q_h^s u||_{L_2(\mathcal{T})} \\ &= ||u||_{H^s(\mathcal{T})} + c \cdot h^{-s} \cdot ||\widetilde{Q}_h (u - Q_h^s u)||_{L_2(\mathcal{T})} \\ &\leq ||u||_{H^s(\mathcal{T})} + \widetilde{c} \cdot h^{-s} \cdot ||u - Q_h^s u||_{L_2(\mathcal{T})} \end{aligned}$$

and the assertion follows from applying the error estimate for Q_h^s in $L_2(\mathcal{T})$.
□

1.5 Quasi Interpolation Operators

To prove the stability estimate (1.76) for globally nonuniform meshes we need to have a projection operator $P_h : H^s(\mathcal{T}) \to V_h \subset H^s(\mathcal{T})$ which is stable in $H^s(\mathcal{T})$ and which admits local error estimates. For this we use quasi interpolation operators as first introduced in [30]. Define

$$(P_h u)(x) = \sum_{k=1}^{M} F_k(u) \cdot \varphi_k(x) \tag{1.77}$$

where $F_k(\cdot) : H^s(\mathcal{T}) \to \mathbb{R}$ are bounded linear functionals. When using Lagrange basis functions one may choose $F_k(u) := u(x_k)$ to get the nodal interpolation operator. It is obvious in this case, that we need stronger regularity assumptions, at least continuity of $u \in H^s(\mathcal{T})$. In what follows we will use local L_2 projections to define the linear functionals $F_k(\cdot)$.

Let Q_h^k denote the Galerkin L_2 projection onto the local trial space $V_h(\omega_k)$ with mesh size $\hat{h}_k$. In particular, for $u \in L_2(\omega_k)$ we define $Q_h^k u \in V_h(\omega_k)$ by

$$\langle Q_h^k u, v_h \rangle_{L_2(\omega_k)} = \langle u, v_h \rangle_{L_2(\omega_k)} \quad \text{for all } v_h \in V_h(\omega_k). \tag{1.78}$$

As in (1.66) we have the stability estimate

$$||Q_h^k u||_{L_2(\omega_k)} \le ||u||_{L_2(\omega_k)} \quad \text{for all } u \in L_2(\omega_k) \tag{1.79}$$

as well as the quasi optimal error estimate

$$||(I - Q_h^k)u||_{L_2(\omega_k)} \le c \cdot \hat{h}_k^s \cdot |u|_{H^s(\omega_k)}, \quad \text{for all } u \in H^s(\omega_k),\ s \in [0,1]. \tag{1.80}$$

Since the mesh is assumed to be locally quasi-uniform, we have the following stability result due to Theorem 1.8:

$$||Q_h^k u||_{H^s(\omega_k)} \le c \cdot ||u||_{H^s(\omega_k)} \quad \text{for all } u \in H^s(\omega_k),\ k = 1, \ldots, M. \tag{1.81}$$

Now we are in a position to define a quasi interpolation or Clement operator by

$$(P_h u)(x) = \sum_{k=1}^{M} (Q_h^k u)(x_k) \cdot \varphi_k(x)\,. \tag{1.82}$$

It is easy to check that P_h is indeed a projection.

Lemma 1.9. [18, 30] *Let $u \in H^s(\mathcal{T})$, $s \in [0,1]$. There exists a positive constant c independent of h such that*

$$||(I - P_h)u||_{L_2(\tau_\ell)} \le c \sum_{k \in J(\ell)} \hat{h}_k^s \cdot |u|_{H^s(\omega_k)} \quad \textit{for } \ell = 1, \ldots, N. \tag{1.83}$$

Moreover,

$$||P_h u||_{H^s(\mathcal{T})} \le c \cdot ||u||_{H^s(\mathcal{T})}. \tag{1.84}$$

Proof. Let τ_ℓ be an arbitrary but fixed finite element and let $\widetilde{k} \in J(\ell)$ be a fixed index. For $x \in \tau_\ell$ we have the representation

$$(P_h u)(x) = (Q_h^{\widetilde{k}} u)(x) + \sum_{k \in J(\ell), k \neq \widetilde{k}} [(Q_h^k u)(x_k) - (Q_h^{\widetilde{k}} u)(x_k)]\varphi_k(x).$$

Let $\sigma = 0, s$. Note that

$$||\varphi_k||_{H^\sigma(\tau_\ell)} \le c \cdot h_\ell^{m/2-\sigma}.$$

Then, using (1.80) it follows that

$$||(I - P_h)u||_{L_2(\tau_\ell)} \le c_1 \cdot \hat{h}_{\widetilde{k}}^s \cdot |u|_{H^s(\omega_{\widetilde{k}})} + c_2 \cdot h_\ell^{m/2} \sum_{k \in J(\ell), k \neq \widetilde{k}} |(Q_h^k u)(x_k) - (Q_h^{\widetilde{k}} u)(x_k)|.$$

Moreover, using (1.81) we get

$$||(I - P_h)u||_{H^s(\tau_\ell)} \le \hat{c}_1 \cdot ||u||_{H^s(\omega_{\widetilde{k}})} + \hat{c}_2 \cdot h_\ell^{m/2-s} \sum_{k \in J(\ell), k \neq \widetilde{k}} |(Q_h^k u)(x_k) - (Q_h^{\widetilde{k}} u)(x_k)|.$$

Now,

$$||v_h||_{L_\infty(\tau_\ell)} \;\leq\; c \cdot h_\ell^{-m/2} \cdot ||v_h||_{L_2(\tau_\ell)} \quad \text{for all } v_h \in V_h,\; \ell = 1, \ldots, N.$$

Thus, (1.80) gives for $x_k \in \tau_\ell$,

$$\begin{aligned}
|(Q_h^k u)(x_k) - (Q_h^{\widetilde{k}} u)(x_k)| &\leq ||Q_h^k u - Q_h^{\widetilde{k}} u||_{L_\infty(\tau_\ell)} \\
&\leq c \cdot h_\ell^{-m/2} \cdot ||Q_h^k u - Q_h^{\widetilde{k}} u||_{L_2(\tau_\ell)} \\
&\leq c \cdot h_\ell^{-m/2} \cdot \Big\{ ||Q_h^k u - u||_{L_2(\tau_\ell)} + ||u - Q_h^{\widetilde{k}} u||_{L_2(\tau_\ell)} \Big\} \\
&\leq c \cdot h_\ell^{-m/2} \cdot \Big\{ \hat{h}_k^s \cdot |u|_{H^s(\omega_k)} + \hat{h}_{\widetilde{k}}^s \cdot |u|_{H^s(\omega_{\widetilde{k}})} \Big\} .
\end{aligned}$$

Hence,

$$||(I - P_h)u||_{L_2(\tau_\ell)} \;\leq\; c \cdot \sum_{k \in J(\ell)} \hat{h}_k^s \cdot |u|_{H^s(\omega_k)}$$

which is (1.83). Moreover,

$$||(I - P_h)u||_{H^s(\tau_\ell)} \;\leq\; \hat{c}_1 \cdot ||u||_{H^s(\omega_{\widetilde{k}})} + \widetilde{c}_2 \cdot h_\ell^{-s} \cdot \sum_{k \in J(\ell)} \hat{h}_k^s \cdot |u|_{H^s(\omega_k)}$$

and therefore, by using (1.53),

$$||(I - P_h)u||_{H^s(\tau_\ell)} \;\leq\; c \cdot \sum_{k \in J(\ell)} ||u||_{H^s(\omega_k)}.$$

For $s = 1$ we now get (1.84) by summing over all elements, while for $s \in (0, 1)$ (1.84) then follows by interpolation. □

Note, that using (1.53) we have the local estimate

$$h_\ell^{-s} \cdot ||(I - P_h)u||_{L_2(\tau_\ell)} \;\leq\; c \cdot \sum_{k \in J(\ell)} |u|_{H^s(\omega_k)} \,.$$

From this we get, by summing up the squares,

$$\sum_{\ell=1}^{N} h_\ell^{-2s} \cdot ||(I - P_h)u||^2_{L_2(\tau_\ell)} \;\leq\; c \cdot ||u||^2_{H^s(\mathcal{T})} \quad \text{for } u \in H^s(\mathcal{T}). \tag{1.85}$$

2

Stability Results

In this chapter we first describe abstract stability results for L_2 projections defined either by Galerkin–Bubnov or Galerkin–Petrov variational formulations. We prove that these operators are bounded in a scale of Sobolev spaces when appropriate assumptions for the underlying finite element spaces are made locally.

For globally quasi-uniform meshes the stability of the Galerkin L_2 projection is a direct consequence of a global inverse inequality, see Theorem 1.8. However, here we are interested in non-structured and nonuniform meshes resulting from adaptive computations.

For nonuniform triangulations in one and two space dimensions the stability of the Galerkin L_2 projection in W_1^p was shown in [34] assuming certain mesh conditions. For a general discussion on the stability of the L_2 projection see, for example, [71]. In [18], new results were given to obtain the H^1 stability of the L_2 projection onto piecewise linear finite element spaces in several space dimensions. This is based on explicit and easily computable mesh criteria. In particular, stability of the L_2 projection in H^1 holds for locally quasi-uniform geometrically refined meshes as long as the volume of neighboring elements does not change too drastically. These results can be extended to prove the stability of the L_2 projection in fractional Sobolev spaces [66]. Then, the mesh criteria depends on the Sobolev index s and disappears for the trivial case $s = 0$. The same techniques can be used to consider the stability of a generalized L_2 projection defined by a Galerkin–Petrov scheme [67], where the test and trial spaces have to satisfy a certain compatibility condition to ensure unique solvability. Here we will give a general approach to deal with almost arbitrary test and trial functions.

Let us consider the generalized L_2 projection $\widetilde{Q}_h$ as defined by the variational problem (1.67). We will prove that the operators

O. Steinbach: LNM 1809, pp. 25–51, 2003.

$$\widetilde{Q}_h : H^s(\mathcal{T}) \to V_h \subset H^s(\mathcal{T}),$$
$$\widetilde{Q}_h^* : \widetilde{H}^{-s}(\mathcal{T}) \to W_h \subset \widetilde{H}^{-s}(\mathcal{T}),$$
$$\Pi_h^s : H^s(\mathcal{T}) \to W_h \subset \widetilde{H}^{-s}(\mathcal{T})$$

are bounded for $s \in [0,1]$ when some appropriate assumptions on the finite element spaces V_h and W_h are satisfied. In particular, the norm of all of these operators is equal. Moreover, there holds also a related stability condition as stated in the following theorem:

Theorem 2.1. *The following statements are equivalent for* $s \in [0,1]$*:*

i.
$$||\widetilde{Q}_h u||_{H^s(\mathcal{T})} \le c_S \cdot ||u||_{H^s(\mathcal{T})} \quad \textit{for all } u \in H^s(\mathcal{T}) \tag{2.1}$$

ii.
$$||\widetilde{Q}_h^* w||_{\widetilde{H}^{-s}(\mathcal{T})} \le c_S \cdot ||w||_{\widetilde{H}^{-s}(\mathcal{T})} \quad \textit{for all } w \in \widetilde{H}^{-s}(\mathcal{T}) \tag{2.2}$$

iii.
$$||\Pi_h^s u||_{\widetilde{H}^{-s}(\mathcal{T})} \le c_S \cdot ||u||_{H^s(\mathcal{T})} \quad \textit{for all } u \in H^s(\mathcal{T}) \tag{2.3}$$

iv.
$$\frac{1}{c_S} \cdot ||u_h||_{H^s(\mathcal{T})} \le \sup_{0 \ne w_h \in W_h} \frac{|\langle u_h, w_h \rangle_{L_2(\mathcal{T})}|}{||w_h||_{\widetilde{H}^{-s}(\mathcal{T})}} \quad \textit{for all } u_h \in V_h \tag{2.4}$$

Proof.
<u>*i.→ii.:*</u> By duality and by using (1.74),

$$\begin{aligned}
||\widetilde{Q}_h^* w||_{\widetilde{H}^{-s}(\mathcal{T})} &= \sup_{0 \ne v \in H^s(\mathcal{T})} \frac{|\langle \widetilde{Q}_h^* w, v \rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} \\
&= \sup_{0 \ne v \in H^s(\mathcal{T})} \frac{|\langle w, \widetilde{Q}_h v \rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} \\
&\le ||w||_{\widetilde{H}^{-s}(\mathcal{T})} \sup_{0 \ne v \in H^s(\mathcal{T})} \frac{||\widetilde{Q}_h v||_{H^s(\mathcal{T})}}{||v||_{H^s(\mathcal{T})}} \le c_S \cdot ||w||_{\widetilde{H}^{-s}(\mathcal{T})}.
\end{aligned}$$

<u>*ii.→i.:*</u> Using similar duality arguments,

$$\begin{aligned}
||\widetilde{Q}_h u||_{H^s(\mathcal{T})} &= \sup_{0 \ne w \in \widetilde{H}^{-s}(\mathcal{T})} \frac{|\langle \widetilde{Q}_h u, w \rangle_{L_2(\mathcal{T})}|}{||w||_{\widetilde{H}^{-s}(\mathcal{T})}} \\
&= \sup_{0 \ne w \in \widetilde{H}^{-s}(\mathcal{T})} \frac{|\langle u, \widetilde{Q}_h^* w \rangle_{L_2(\mathcal{T})}|}{||w||_{\widetilde{H}^{-s}(\mathcal{T})}} \\
&\le ||u||_{H^s(\mathcal{T})} \sup_{0 \ne w \in \widetilde{H}^{-s}(\mathcal{T})} \frac{||\widetilde{Q}_h^* w||_{\widetilde{H}^{-s}(\mathcal{T})}}{||w||_{\widetilde{H}^{-s}(\mathcal{T})}} \le c_S \cdot ||u||_{H^s(\mathcal{T})}.
\end{aligned}$$

<u>*i.→iii.*:</u> By duality,

$$\begin{aligned}||\Pi_h^s u||_{\widetilde{H}^{-s}(\mathcal{T})} &= \sup_{0\neq v\in H^s(\mathcal{T})} \frac{|\langle \Pi_h^s u, v\rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} \\ &= \sup_{0\neq v\in H^s(\mathcal{T})} \frac{|\langle \Pi_h^s u, \widetilde{Q}_h v\rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} \\ &= \sup_{0\neq v\in H^s(\mathcal{T})} \frac{|\langle u, \widetilde{Q}_h v\rangle_{H^s(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} \\ &\le ||u||_{H^s(\mathcal{T})} \sup_{0\neq v\in H^s(\mathcal{T})} \frac{||\widetilde{Q}_h v||_{H^s(\mathcal{T})}}{||v||_{H^s(\mathcal{T})}} \le c_S \cdot ||u||_{H^s(\mathcal{T})}.\end{aligned}$$

<u>*iii.→iv.*:</u> Using $w_h^* = \Pi_h^s u_h \in W_h$,

$$\begin{aligned}\sup_{0\neq w_h\in W_h} \frac{|\langle u_h, w_h\rangle_{L_2(\mathcal{T})}|}{||w_h||_{\widetilde{H}^{-s}(\mathcal{T})}} &\ge \frac{|\langle u_h, \Pi_h^s u_h\rangle_{L_2(\mathcal{T})}|}{||\Pi_h^s u_h||_{\widetilde{H}^{-s}(\mathcal{T})}} \\ &= \frac{||u_h||^2_{H^s(\mathcal{T})}}{||\Pi_h^s u_h||_{\widetilde{H}^{-s}(\mathcal{T})}} \ge \frac{1}{c_S} \cdot ||u_h||_{H^s(\mathcal{T})}.\end{aligned}$$

<u>*iv.→i.*:</u> For $u_h = \widetilde{Q}_h u$,

$$\begin{aligned}\frac{1}{c_S} \cdot ||\widetilde{Q}_h u||_{H^s(\mathcal{T})} &\le \sup_{0\neq w_h\in W_h} \frac{|\langle \widetilde{Q}_h u, w_h\rangle_{L_2(\mathcal{T})}|}{||w_h||_{\widetilde{H}^{-s}(\mathcal{T})}} \\ &= \sup_{0\neq w_h\in W_h} \frac{|\langle u, w_h\rangle_{L_2(\mathcal{T})}|}{||w_h||_{\widetilde{H}^{-s}(\mathcal{T})}} \le ||u||_{H^s(\mathcal{T})}. \quad \square\end{aligned}$$

In Theorem 1.8 we proved the stability estimate (2.1) when assuming a global inverse inequality. However, for nonuniform meshes, in particular, for meshes resulting from some adaptive refinement procedures, Theorem 1.8 is not applicable. However, in what follows we will prove the equivalent stability of Π_h^s assuming appropriate sufficient conditions locally. For this, we first introduce local scaling matrices as follows. For some arbitrary but fixed $s \in [0, 1]$ we define the diagonal matrix

$$H_\ell = \operatorname{diag}\left(\hat{h}_k^s\right)_{k=1}^{M_\ell} \tag{2.5}$$

where the $\hat{h}_k$ are local mesh parameters associated with a basis function φ_k, $k \in J(\ell)$. We assume that the mesh is locally quasi uniform and that (1.53) is satisfied.

Now we are in a position to formulate a sufficient condition which is essentially needed in the proof of the next theorem. For the local Gram matrices $\widetilde{G}_\ell$ defined by (1.61) we make the following assumption:

Assumption 2.1 *We assume that there exists a positive constant* $c_0 > 0$ *such that*

$$(H_\ell \widetilde{G}_\ell^\top H_\ell^{-1} \underline{x}_\ell, \underline{x}_\ell) \geq c_0 \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \quad \textit{for all}\, \underline{x}_\ell \in \mathbb{R}^{M_\ell} \tag{2.6}$$

holds uniformly for all $\ell = 1, \ldots, N$.

Note that for a globally quasi-uniform mesh we may chose $\hat{h}_k = h$ yielding $H_\ell \widetilde{G}_\ell^\top H_\ell^{-1} = \widetilde{G}_\ell^\top$. In this case, Assumption 2.1 coincides with Assumption 1.2. If we define the symmetric matrix

$$G_\ell^S := \frac{1}{2} \cdot \left[H_\ell \widetilde{G}_\ell^\top H_\ell^{-1} + H_\ell^{-1} \widetilde{G}_\ell H_\ell \right], \tag{2.7}$$

(2.6) is equivalent to

$$(G_\ell^S \underline{x}_\ell, \underline{x}_\ell) \geq c_0 \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \quad \text{for all}\, \underline{x}_\ell \in \mathbb{R}^{M_\ell}. \tag{2.8}$$

Theorem 2.2. *Let Assumption 2.1 be satisfied for some* $s \in (0, 1]$. *Then,*

$$||\Pi_h^s u||_{\widetilde{H}^{-s}(\mathcal{T})} \leq c_S \cdot ||u||_{H^s(\mathcal{T})} \quad \textit{for all}\, u \in H^s(\mathcal{T}). \tag{2.9}$$

The proof of this theorem is based on the following lemma. A similar estimate was used in [19] to construct spectrally equivalent multilevel preconditioners in finite element methods in the case of globally quasi-uniform meshes.

Lemma 2.3. *Let Assumption 2.1 be satisfied and* $\varphi_k \in V_h$ *for* $k = 1, \ldots, M$. *Then,*

$$\sum_{\ell=1}^{N} h_\ell^{2s} \cdot ||w_h||^2_{L_2(\tau_\ell)} \leq c \cdot \sum_{k=1}^{M} \left[\frac{\langle w_h, \varphi_k \rangle_{L_2(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2 \tag{2.10}$$

for all $w_h \in W_h$ *with a positive constant.*

Proof. Using Assumption 1.1 for $V_h = W_h$ we have

$$\begin{aligned} \sum_{\ell=1}^{N} h_\ell^{2s} \cdot ||w_h||^2_{L_2(\tau_\ell)} &\leq c \cdot \sum_{\ell=1}^{N} h_\ell^{2s} \cdot \sum_{k \in J(\ell)} w_k^2 \cdot ||\psi_k||^2_{L_2(\tau_\ell)} \\ &= c \cdot \sum_{k=1}^{M} w_k^2 \sum_{\ell \in I(k)} h_\ell^{2s} \cdot ||\psi_k||^2_{L_2(\tau_\ell)} = c \cdot \sum_{k=1}^{M} w_k^2 \gamma_k^2 \end{aligned}$$

with

$$\gamma_k := \sqrt{\sum_{\ell \in I(k)} h_\ell^{2s} \cdot ||\psi_k||^2_{L_2(\tau_\ell)}} \quad \text{for } k = 1, \ldots, M.$$

Setting $x_k := \gamma_k w_k$ this gives

$$\sum_{\ell=1}^{N} h_\ell^{2s} \cdot ||w_h||^2_{L_2(\tau_\ell)} \;\le\; c \cdot ||\underline{x}||_2^2. \tag{2.11}$$

On the other hand,

$$\sum_{k=1}^{M} \left[\frac{\langle w_h, \varphi_k \rangle_{L_2(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2 = \sum_{k=1}^{M} \left[\sum_{j=1}^{M} w_j \frac{\langle \psi_j, \varphi_k \rangle_{L_2(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2$$

$$= \sum_{k=1}^{M} \left[\sum_{j=1}^{M} x_j \frac{\langle \psi_j, \varphi_k \rangle_{L_2(\mathcal{T})}}{\gamma_j \cdot ||\varphi_k||_{H^s(\omega_k)}} \right]^2 \;=\; ||A\underline{x}||_2^2 \tag{2.12}$$

with a matrix A given by

$$A \;:=\; D_\varphi^{-1} \widetilde{G}_h^\top D_\gamma^{-1}, \quad D_\varphi \;:=\; \operatorname{diag}(||\varphi_k||_{H^s(\omega_k)}), \quad D_\gamma \;:=\; \operatorname{diag}(\gamma_k)\,.$$

Let $\bar{G}_h := H \widetilde{G}_h^\top H^{-1}$. For any $\underline{u} \in \mathbb{R}^M$ we define

$$v_h \;:=\; \sum_{k=1}^{M} \hat{h}_k^{-s} u_k \varphi_k \in V_h, \qquad w_h \;:=\; \sum_{k=1}^{M} \hat{h}_k^{s} u_k \psi_k \in W_h.$$

Then, using Assumption 2.1,

$$(\bar{G}_h \underline{u}, \underline{u}) = (\widetilde{G}_h^\top H^{-1} \underline{u}, H\underline{u}) \;=\; \langle w_h, v_h \rangle_{L_2(\mathcal{T})} \;=\; \sum_{\ell=1}^{N} \langle w_h, v_h \rangle_{L_2(\tau_\ell)}$$

$$= \sum_{\ell=1}^{N} (H_\ell \widetilde{G}_\ell^\top H_\ell^{-1} \underline{u}_\ell, \underline{u}_\ell) \;\ge\; c_0 \cdot \sum_{\ell=1}^{N} (D_\ell \underline{u}_\ell, \underline{u}_\ell) \;=\; c_0 \cdot (D_h \underline{u}, \underline{u}).$$

Since all entries of D_h are strictly positive, we can define

$$D_h^{1/2} := \operatorname{diag}\left(||\varphi_k||_{L_2(\mathcal{T})} \right).$$

From

$$c_0 \cdot ||D_h^{1/2} \underline{u}||_2^2 = c_0 \cdot (D_h \underline{u}, \underline{u}) \;\le\; (\bar{G}_h \underline{u}, \underline{u})$$

$$= (D_h^{-1/2} \bar{G}_h \underline{u}, D_h^{1/2} \underline{u}) \;\le\; ||D_h^{-1/2} \bar{G}_h \underline{u}||_2 ||D_h^{1/2} \underline{u}||_2$$

we conclude that

$$c_0 \cdot ||D_h^{1/2} \underline{u}||_2 \;\le\; ||D_h^{-1/2} \bar{G}_h \underline{u}||_2 \quad \text{for all } \underline{u} \in \mathbb{R}^M.$$

Taking $\underline{v} := D_\gamma \underline{u}$ this is equivalent to

$$c_0 \cdot ||D_h^{1/2} D_\gamma^{-1} \underline{v}||_2 \;\leq\; ||D_h^{-1/2} D_\varphi D_\varphi^{-1} \bar{G}_h D_\gamma^{-1} \underline{v}||_2 \;=\; ||D_h^{-1/2} D_\varphi \widetilde{A} \underline{v}||_2$$

for all $\underline{v} \in \mathbb{R}^M$ using $\widetilde{A} := D_\varphi^{-1} \bar{G}_h D_\gamma^{-1}$. The ratio of the diagonal entries satisfies

$$\frac{D_h^{1/2}[k,k]}{D_\gamma[k,k]} \;=\; \frac{\sqrt{\sum\limits_{\ell \in I(k)} ||\varphi_k||^2_{L_2(\tau_\ell)}}}{\sqrt{\sum\limits_{\ell \in I(k)} h_\ell^{2s} ||\varphi_k||^2_{L_2(\tau_\ell)}}} \;\geq\; c \cdot \frac{1}{\hat{h}_k^s}$$

and

$$\frac{D_\varphi[k,k]}{D_h^{1/2}[k,k]} \;=\; \frac{||\varphi_k||_{H^s(\omega_k)}}{||\varphi_k||_{L_2(\mathcal{T})}} \;\leq\; c \cdot \frac{1}{\hat{h}_k^s}$$

for all $k = 1, \ldots, M$ due to the inverse inequality locally. Thus,

$$c \cdot ||H^{-1} \underline{v}||_2 \;\leq\; ||H^{-1} \widetilde{A} \underline{v}||_2 \quad \text{for all } \underline{v} \in \mathbb{R}^M .$$

Taking $\underline{x} := H^{-1} \underline{v}$ above gives

$$\begin{aligned} c \cdot ||\underline{x}||_2 \;\leq\; ||H^{-1} \widetilde{A} H \underline{x}||_2 &= ||H^{-1} D_\varphi^{-1} H \widetilde{G}_h^\top H^{-1} D_\gamma^{-1} H \underline{x}||_2 \\ &= ||D_\varphi^{-1} \widetilde{G}_h^\top D_\gamma^{-1} \underline{x}||_2 \;=\; ||A \underline{x}||_2 \end{aligned}$$

for all $\underline{x} \in \mathbb{R}^M$ since D_φ, D_γ and H are diagonal matrices. Combining this with (2.11) and (2.12) completes the proof. □

For the next result see also [39, Theorem 1.4.3.1].

Lemma 2.4. *Let* $s \in [0,1]$. *Then,*

$$||\varphi_k||_{H^s(\mathbb{R}^m)} \;\leq\; c \cdot ||\varphi_k||_{H^s(\omega_k)} \quad \textit{for all } \varphi_k \in V_h. \tag{2.13}$$

Proof. Since the assertion is obviously satisfied for $s = 0, 1$ it is sufficient to consider the semi-norms only. Let $B_{r_k}(x_k)$ the smallest ball which circumscribes ω_k with mid point x_k and radius $r_k > 0$. Let

$$\hat{\omega_k} \;:=\; \{x \in \mathbb{R}^m \;:\; |x - x_k| \leq 2 r_k\} \;.$$

Then, for $x \in \omega_k$ and $y \in \mathbb{R}^m \backslash \hat{\omega}_k$,

$$|x - y| \;\geq\; |y - x_k| - |x - x_k| \;\geq\; |y - x_k| - r_k \,.$$

Now, by definition,

$$\begin{aligned} |\varphi_k|^2_{H^s(\mathbb{R}^n)} &= \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^m} \frac{|\varphi_k(x) - \varphi_k(y)|^2}{|x-y|^{m+2s}} dx dy \\ &= \int\limits_{\omega_k} \int\limits_{\omega_k} \frac{|\varphi_k(x) - \varphi_k(y)|^2}{|x-y|^{m+2s}} dx dy \;+\; 2 \cdot \int\limits_{\omega_k} \int\limits_{\mathbb{R}^m \backslash \omega_k} \frac{|\varphi_k(x)|^2}{|x-y|^{m+2s}} dx dy \\ &= |\varphi_k|^2_{H^s(\omega_k)} \;+\; 2 \cdot \int\limits_{\omega_k} \int\limits_{\mathbb{R}^m \backslash \omega_k} \frac{|\varphi_k(x)|^2}{|x-y|^{m+2s}} dx dy \,. \end{aligned}$$

For the remaining term we have

$$\int_{\omega_k}\int_{\mathbb{R}^m\setminus\omega_k} \frac{|\varphi_k(x)|^2}{|x-y|^{m+2s}}dxdy$$

$$= \int_{\omega_k}\int_{\hat\omega_k\setminus\omega_k} \frac{|\varphi_k(x)|^2}{|x-y|^{m+2s}}dxdy + \int_{\omega_k}\int_{\mathbb{R}^m\setminus\hat\omega_k} \frac{|\varphi_k(x)|^2}{|x-y|^{m+2s}}dxdy \ .$$

The first summand can be estimated by

$$\int_{\omega_k}\int_{\hat\omega_k\setminus\omega_k} \frac{|\varphi_k(x)|^2}{|x-y|^{m+2s}}dxdy = \int_{\omega_k}\int_{\hat\omega_k\setminus\omega_k} \frac{|\varphi_k(x)-\varphi_k(y)|^2}{|x-y|^{m+2s}}dxdy$$

$$\leq \int_{\hat\omega_k}\int_{\hat\omega_k} \frac{|\varphi_k(x)-\varphi_k(y)|^2}{|x-y|^{m+2s}}dxdy$$

$$= |\varphi_k|^2_{H^s(\hat\omega_k)} \ \leq \ c\cdot|\varphi_k|^2_{H^s(\omega_k)}.$$

The second summand can be estimated as

$$\int_{\omega_k}\int_{\mathbb{R}^m\setminus\hat\omega_k} \frac{|\varphi_k(x)|^2}{|x-y|^{m+2s}}dxdy \leq \int_{\omega_k}|\varphi_k(x)|^2dx \int_{\mathbb{R}^m\setminus\hat\omega_k} \frac{1}{(|y-x_k|-r_k)^{m+2s}}dy$$

$$\leq c\cdot\frac{1}{s}\cdot r_k^{-2s}\cdot||\varphi_k||^2_{L_2(\omega_k)} \ \leq \ \tilde c\cdot\frac{1}{s}\cdot r_k^{m-2s} \ .$$

Here we used, by explicit computations,

$$\int_{\mathbb{R}\setminus\hat\omega_k} \frac{dy}{(|y-x_k|-r_k)^{1+2s}} = 2\cdot\int_{2r_k}^{\infty} \frac{dr}{(r-r_k)^{1+2s}} \ = \ \frac{1}{s}\cdot r_k^{-2s},$$

$$\int_{\mathbb{R}^2\setminus\hat\omega_k} \frac{dy}{(|y-x_k|-r_k)^{2+2s}} = \int_0^{2\pi}\int_{2r_k}^{\infty} \frac{rdrd\varphi}{(r-r_k)^{2+2s}} \ = \ \pi\cdot\frac{1+4s}{s(1+2s)}\cdot r_k^{-2s} \ ,$$

$$\int_{\mathbb{R}^3\setminus\hat\omega_k} \frac{dy}{(|y-x_k|-r_k)^{3+2s}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_{2r_k}^{\infty} \frac{r^2\cos\theta drd\varphi d\theta}{(r-r_k)^{3+2s}}$$

$$= 4\pi\int_{2r_k}^{\infty} \frac{r^2}{(r-r_k)^{3+2s}}dr \ = \ 2\pi\cdot\frac{8s^2+8s+1}{s(1+2s)(1+s)}\cdot r_k^{-2s}.$$

On the other hand we have

$$|\varphi_k|^2_{H^s(\omega_k)} \ = \int_{\omega_k}\int_{\omega_k} \frac{|\varphi_k(x)-\varphi_k(y)|^2}{|x-y|^{m+2s}}dxdy \ \sim \ \frac{1}{s}\cdot\hat h_k^{m-2s} \ .$$

Using that the mesh is locally quasi-uniform, we have $r_k \sim \hat{h}_k$ and the assertion follows. □

Lemma 2.5. *For $v \in H^s(\mathcal{T})$ with $s \in (0,1]$ and $\varphi_k \in V_h$, $k = 1, \ldots, M$, there holds the estimate*

$$\sum_{k=1}^{M} \left[\frac{\langle v, \varphi_k \rangle_{H^s(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2 \leq c \cdot ||v||^2_{H^s(\mathcal{T})}. \tag{2.14}$$

Proof. For $s = 0, 1$ we have

$$\begin{aligned}
\sum_{k=1}^{M} \left[\frac{\langle v, \varphi_k \rangle_{H^s(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2 &= \sum_{k=1}^{M} \left[\frac{\langle v, \varphi_k \rangle_{H^s(\omega_k)}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2 \leq \sum_{k=1}^{M} ||v||^2_{H^s(\omega_k)} \\
&= \sum_{\ell=1}^{N} \sum_{k \in J(\ell)} ||v||^2_{H^s(\tau_\ell)} \\
&\leq \max_{\ell=1,\ldots,N} \{\# J(\ell)\} \cdot ||v||^2_{H^s(\mathcal{T})}
\end{aligned}$$

and therefore (2.14). For $s \in (0,1)$ the inner product in $H^s(\mathcal{T})$ can be described by

$$\langle u, v \rangle_{H^s(\mathcal{T})} = \langle u, v \rangle_{L_2(\mathcal{T})} + b(u, v)$$

with

$$b(u,v) = \int_{\mathcal{T}} \int_{\mathcal{T}} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{m+2s}} dxdy.$$

Therefore,

$$\left[\frac{\langle v, \varphi_k \rangle_{H^s(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2 \leq 2 \cdot ||v||^2_{L_2(\omega_k)} + 2 \cdot \left[\frac{b(v, \varphi_k)}{||\varphi_k||_{H^s(\omega_k)}} \right]^2.$$

Using

$$\begin{aligned}
b(v, \varphi_k) = &\int_{\omega_k} \int_{\omega_k} \frac{[v(x) - v(y)][\varphi_k(x) - \varphi_k(y)]}{|x - y|^{m+2s}} dxdy \\
&+ 2 \cdot \int_{\omega_k} \int_{\mathcal{T} \setminus \omega_k} \frac{[v(x) - v(y)]\varphi_k(x)}{|x - y|^{m+2s}} dxdy
\end{aligned}$$

and the Cauchy–Schwarz inequality we get

$$[b(v, \varphi)]^2 \leq 2 \cdot \left\{ |v|^2_{H^s(\omega_k)} |\varphi_k|^2_{H^s(\omega_k)} + 4\mu_k \int_{\omega_k} \int_{\mathcal{T} \setminus \omega_k} \frac{[v(x) - v(y)]^2}{|x - y|^{m+2s}} dxdy \right\}$$

with

$$\mu_k = \int\limits_{\omega_k} \int\limits_{\mathcal{T}\setminus\omega_k} \frac{[\varphi_k(x)]^2}{|x-y|^{m+2s}} dxdy \;=\; \int\limits_{\omega_k} \int\limits_{\mathcal{T}\setminus\omega_k} \frac{[\varphi_k(x)-\varphi_k(y)]^2}{|x-y|^{m+2s}} dxdy$$

$$\leq \int\limits_{\mathcal{T}} \int\limits_{\mathcal{T}} \frac{[\varphi_k(x)-\varphi_k(y)]^2}{|x-y|^{m+2s}} dxdy \;=\; |\varphi_k|^2_{H^s(\mathcal{T})} \;\leq\; c \cdot ||\varphi_k||^2_{H^s(\omega_k)}.$$

Hence, (2.14) follows by summing up over all $k = 1, \ldots, M$. □

Now we are able to give the proof of Theorem 2.2:

Proof of Theorem 2.2. For $v \in H^s(\mathcal{T})$ let $P_h v \in V_h$ satisfying the stability estimate (1.84) and the error estimate (1.85). Then, by duality, using definition (1.75) and (1.84),

$$||\Pi_h^s u||_{\widetilde{H}^{-s}(\mathcal{T})} = \sup_{0\neq v\in H^s(\mathcal{T})} \frac{|\langle \Pi_h^s u, v\rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}}$$

$$= \sup_{0\neq v\in H^s(\mathcal{T})} \left\{ \frac{|\langle \Pi_h^s u, P_h v\rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} + \frac{|\langle \Pi_h^s u, (I-P_h)v\rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} \right\}$$

$$= \sup_{0\neq v\in H^s(\mathcal{T})} \left\{ \frac{|\langle u, P_h v\rangle_{H^s(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} + \frac{|\langle \Pi_h^s u, (I-P_h)v\rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}} \right\}$$

$$\leq c \cdot ||u||_{H^s(\mathcal{T})} + \sup_{0\neq v\in H^s(\mathcal{T})} \frac{|\langle \Pi_h^s u, (I-P_h)v\rangle_{L_2(\mathcal{T})}|}{||v||_{H^s(\mathcal{T})}}.$$

For the second summand we use the Cauchy–Schwarz inequality to obtain

$$|\langle \Pi_h^s u, (I-P_h)v\rangle_{L_2(\mathcal{T})} \leq \sum_{\ell=1}^N |\langle \Pi_h^s u, (I-P_h)v\rangle_{L_2(\tau_\ell)}|$$

$$\leq \sum_{\ell=1}^N ||\Pi_h^s u||_{L_2(\tau_\ell)} ||(I-P_h)v||_{L_2(\tau_\ell)}$$

$$\leq \left(\sum_{\ell=1}^N h_\ell^{2s} \cdot ||\Pi_h^s u||^2_{L_2(\tau_\ell)} \right)^{1/2} \left(\sum_{\ell=1}^N h_\ell^{-2s} \cdot ||(I-P_h)v||^2_{L_2(\tau_\ell)} \right)^{1/2}$$

$$\leq c \cdot \left(\sum_{\ell=1}^N h_\ell^{2s} \cdot ||\Pi_h^s u||^2_{L_2(\tau_\ell)} \right)^{1/2} ||v||_{H^s(\mathcal{T})}$$

due to (1.85). Now, the assertion follows from Lemma 2.3, definition (1.75) and Lemma 2.5. Indeed,

$$\sum_{\ell=1}^{N} h_\ell^{2s} \cdot ||\Pi_h^s u||^2_{L_2(\tau_\ell)} \leq c \cdot \sum_{k=1}^{M} \left[\frac{\langle \Pi_h^s u, \varphi_k \rangle_{L_2(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2$$

$$= c \cdot \sum_{k=1}^{M} \left[\frac{\langle u, \varphi_k \rangle_{H^s(\mathcal{T})}}{||\varphi_k||_{H^s(\omega_k)}} \right]^2 \leq c \cdot ||u||^2_{H^s(\mathcal{T})}. \quad \square$$

The proof of Theorem 2.2 was essentially based on Assumption 2.1. In what follows we will consider several situations when (2.8) and therefore (2.6) is satisfied. We first consider the Galerkin L_2 projection Q_h defined by (1.64) using piecewise linear test and trial functions. Then we investigate the stability of the generalized L_2 projection $\widetilde{Q}_h$ when using piecewise constant test functions which are defined on the dual mesh. For these two cases we obtain explicit and computable conditions by computing the minimal eigenvalue of G_ℓ^S. Then, for $m = 1$ we give discuss some numerical results concerning the stability of Q_h when using higher order polynomials to define the finite element space V_h. Finally we discuss the use of biorthogonal basis functions as introduced in [73]. In this case, (2.1) is trivially satisfied and no additional restrictions appear.

2.1 Piecewise Linear Elements

First we consider the standard L_2 projection Q_h with $V_h = W_h$ using piecewise linear basis functions defined with respect to a triangulation $\mathcal{T}_N$, see [18] for $s = 1$ and [66] for $s \in (0, 1)$. Here, a finite element τ_ℓ is either an interval ($m = 1$), a triangle ($m = 2$) or a tetrahedron ($m = 3$). Then,

$$V_h := \operatorname{span}\{\varphi_k\}_{k=1}^M \subset H^1(\mathcal{T}) \quad \text{with } \varphi_k(x_j) = \delta_{kj}. \tag{2.15}$$

The local Gram matrices (1.46) are given by

$$G_\ell = \frac{1}{(m+1)(m+2)} \cdot \Delta_\ell \cdot (1 + \delta_{ij})_{i,j=1}^{m+1}$$

with diagonal parts

$$D_\ell = \frac{2}{(m+1)(m+2)} \cdot \Delta_\ell \cdot I_{m+1}.$$

Let $\underline{e}_{m+1} \in \mathbb{R}^{m+1}$ denote the vector with $e_k = 1$ for all $k = 1, \ldots, m+1$. Then,

$$G_\ell = \frac{1}{(m+1)(m+2)} \cdot \Delta_\ell \cdot \left[I_{m+1} + \underline{e}_{m+1} \, \underline{e}_{m+1}^\top \right].$$

Therefore, all eigenvalues of the generalized eigenvalue problem

$$G_\ell \underline{x} = \lambda_i \cdot D_\ell \underline{x} \quad \text{for } \underline{x} \in \mathbb{R}^{m+1}$$

are given by

$$\lambda_1 = \frac{1}{2}(2+m), \quad \lambda_i = \frac{1}{2} \quad \text{for } i = 2, \ldots, m+1. \tag{2.16}$$

Hence we have

$$\frac{1}{2} \cdot (D_\ell \underline{x}, \underline{x}) \leq (G_\ell \underline{x}, \underline{x}) \leq \frac{1}{2}(2+m) \cdot (D_\ell \underline{x}, \underline{x}) \quad \text{for all } \underline{x} \in \mathbb{R}^{m+1} \tag{2.17}$$

and therefore Assumption 1.1 is satisfied with

$$c_1^G = \frac{1}{2}, \qquad c_2^G = \frac{1}{2}(2+m).$$

Next we have to check Assumption 2.1 locally. For this we compute the symmetric matrix G_ℓ^S, see (2.7),

$$G_\ell^S := \frac{1}{2} \cdot \frac{1}{(m+1)(m+2)} \cdot \Delta_\ell \cdot A_{m+1} \tag{2.18}$$

with

$$A_{m+1} := H_\ell[I_{m+1} + \underline{e}_{m+1}\underline{e}_{m+1}^\top]H_\ell^{-1} + H_\ell^{-1}[I_{m+1} + \underline{e}_{m+1}\underline{e}_{m+1}^\top]H_\ell.$$

Hence to show (2.8) it is sufficient to consider the eigenvalues of the matrix A_{m+1}. Let

$$\underline{a} := (\hat{h}_{k_1}^s, \ldots, \hat{h}_{k_{m+1}}^s)^\top, \quad \underline{b} := (\frac{1}{\hat{h}_{k_1}^s}, \ldots, \frac{1}{\hat{h}_{k_{m+1}}^s})^\top.$$

Note that

$$\underline{a} \cdot \underline{b} = m+1.$$

Obviously,

$$A_{m+1} = 2 \cdot I_{m+1} + \underline{a}\,\underline{b}^\top + \underline{b}\,\underline{a}^\top$$

is a rank 2 perturbation of the diagonal matrix $2\,I_{m+1}$. To compute the eigenvalues of

$$\widetilde{A}_{m+1} := \underline{a}\,\underline{b}^\top + \underline{b}\,\underline{a}^\top$$

we first assume that $\underline{a}$ and $\underline{b}$ are linear independent. Any eigenvector $\underline{x}$ of $\widetilde{A}_{m+1}$ can be written as

$$\underline{x} = \alpha \underline{a} + \beta \underline{b}.$$

Then, the eigenvalue problem

$$\widetilde{A}_{m+1} \underline{x} = \widetilde{\lambda} \underline{x}$$

is equivalent to the matrix eigenvalue problem

$$\begin{pmatrix} m+1 & \underline{b}^\top \underline{b} \\ \underline{a}^\top \underline{a} & m+1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \widetilde{\lambda} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Hence the nonzero eigenvalues of $\widetilde{A}_{m+1}$ are given by

$$\widetilde{\lambda}_{1/2} = m + 1 \pm \sqrt{(\underline{a}^\top \underline{a})(\underline{b}^\top \underline{b})}.$$

The eigenvalues of A_{m+1} are now given by

$$\lambda_{1/2} = 3 + m \pm \sqrt{\sum_{k \in J(\ell)} \hat{h}_k^{2s} \sum_{k \in J(\ell)} \hat{h}_k^{-2s}}, \quad \lambda_3 = \ldots = \lambda_{m+1} = 2. \qquad (2.19)$$

Note that this remains true when $\underline{a}$ and $\underline{b}$ are linear dependent. Then we have $\hat{h}_k = h$ for all $k \in J(\ell)$ and

$$\lambda_1 = 4, \quad \lambda_2, \ldots, \lambda_{m+1} = 2.$$

This corresponds to the special case of a globally quasi-uniform mesh.

To ensure that the matrix A_{m+1} is positive definite we have to require

$$\lambda_0 := \min_{\ell=1,\ldots,N} \left\{ 3 + m - \sqrt{\sum_{k \in J(\ell)} \hat{h}_k^{2s} \sum_{k \in J(\ell)} \hat{h}_k^{-2s}} \right\} > 0 \qquad (2.20)$$

uniformly for all $\ell = 1, \ldots, N$. Then, (2.8) holds with $c_0 = \frac{1}{2}\lambda_0$.

Remark 2.6. In fact, (2.20) is a local mesh condition to be satisfied. Since $\hat{h}_k$ is equivalent to the mesh size of all neighboring elements, (2.20) describes a bound for the ratio of the volume of neighboring elements. Note that for a globally quasi-uniform mesh we can set $\hat{h}_k = h$ to get $\lambda_0 = 2$. Hence we have stability as already proved for globally quasi-uniform meshes, see Theorem 1.8.

Remark 2.7. Taking the limit $s \to 0$ in (2.20) we get $\lambda_0 = 2$ implying the stability of the L_2 projection in $L_2(\mathcal{T})$ without any further condition. This means, that the mesh condition (2.20) reflects the dependency of the Sobolev index $s \in [0, 1]$ in a natural way.

Next we will discuss a few examples illustrating the theoretical results stated until now.

As a first example, for $m = 1$, we consider the interval $[0, 1]$ which is decomposed into N elements τ_ℓ. For $q > 1$ we define a geometrically refined mesh by

$$h := \frac{q-1}{q^N - 1}, \quad h_\ell := hq^{\ell-1}, \quad x_1 := 0, \quad x_{\ell+1} := x_\ell + h_\ell \quad \text{for } \ell = 1, \ldots, N. \qquad (2.21)$$

For $m = 1$ the mesh condition (2.20) reads

$$\lambda_0 := \min_{\ell=1,\ldots,N} \left\{ 4 - \frac{\hat{h}_\ell}{\hat{h}_{\ell+1}} - \frac{\hat{h}_{\ell+1}}{\hat{h}_\ell} \right\} > 0 \,.$$

In what follows we will demonstrate, that the specific definition of the associated mesh size $\hat{h}_k$ may not have any influence of the resulting stability condition (2.20).

i. Averaging of local mesh sizes:

$$\hat{h}_\ell = \begin{cases} h & \text{for } \ell = 1, \\ \frac{1}{2} h q^{\ell-2}(1+q) & \text{for } \ell = 2, \ldots, N, \\ h q^{N-1} & \text{for } \ell = N+1. \end{cases} \tag{2.22}$$

Then,

$$\frac{\hat{h}_\ell}{h_{\ell-1}} = \begin{cases} \frac{1}{2}(1+q) & \text{for } \ell = 2, \ldots, N, \\ 1 & \text{for } \ell = N+1, \end{cases}$$

$$\frac{\hat{h}_\ell}{h_\ell} = \begin{cases} 1 & \text{for } \ell = 1, \\ \frac{1}{2}(1+\frac{1}{q}) & \text{for } \ell = 2, \ldots, N. \end{cases}$$

Hence we have for $k = 1, \ldots, N+1$

$$\frac{1}{2}(1+\frac{1}{q}) \cdot h_\ell \le \hat{h}_k \le \frac{1}{2}(1+q) \cdot h_\ell \quad \text{for all } \ell \in I(k),$$

and therefore (1.53) is satisfied. Now,

$$\frac{\hat{h}_\ell}{\hat{h}_{\ell+1}} = \begin{cases} \frac{2}{1+q} & \text{for } \ell = 1, \\ \frac{1}{q} & \text{for } \ell = 2, \ldots, N-1, \\ \frac{1+q}{2q} & \text{for } \ell = N. \end{cases}$$

The mesh condition (2.20) then reads

$$\lambda_0 = \min \left\{ 4 - \frac{2}{1+q} - \frac{1+q}{2}, 4 - \frac{1}{q} - q, 4 - \frac{1+q}{2q} - \frac{2q}{1+q} \right\} > 0$$

from which we get

$$q < 2 + \sqrt{3}.$$

ii. Mass of basis functions:

$$\hat{h}_\ell = \begin{cases} \frac{1}{3}h & \text{for } \ell = 1, \\ \frac{1}{3}hq^{\ell-2}(1+q) & \text{for } \ell = 2, \ldots, N, \\ \frac{1}{3}hq^{N-1} & \text{for } \ell = N+1. \end{cases}$$

Then,

$$\frac{\hat{h}_\ell}{h_{\ell-1}} = \begin{cases} \frac{1}{3}(1+q) & \text{for } \ell = 2, \ldots, N, \\ \frac{1}{3} & \text{for } \ell = N+1, \end{cases}$$

$$\frac{\hat{h}_\ell}{h_\ell} = \begin{cases} \frac{1}{3} & \text{for } \ell = 1, \\ \frac{1}{3}(1+\frac{1}{q}) & \text{for } \ell = 2, \ldots, N. \end{cases}$$

Hence we have for $k = 1, \ldots, N+1$

$$\frac{1}{3} \cdot h_\ell \leq \hat{h}_k \leq \frac{1}{3}(1+q) \cdot h_\ell \quad \text{for all } \ell \in I(k).$$

Now,

$$\frac{\hat{h}_\ell}{\hat{h}_{\ell+1}} = \begin{cases} \frac{1}{1+q} & \text{for } \ell = 1, \\ \frac{1}{q} & \text{for } \ell = 2, \ldots, N-1, \\ \frac{1+q}{q} & \text{for } \ell = N. \end{cases}$$

The mesh condition (2.20) then reads

$$\lambda_0 = \min\left\{4 - \frac{1}{1+q} - (1+q), 4 - \frac{1}{q} - q, 4 - \frac{1+q}{q} - \frac{q}{1+q}\right\} > 0$$

from which we get

$$q < 1 + \sqrt{3}.$$

iii. Minimal distance of associated vertices:

$$\hat{h}_\ell = \begin{cases} h & \text{for } \ell = 1, \\ hq^{\ell-1} & \text{for } \ell = 2, \ldots, N, \\ hq^{N-1} & \text{for } \ell = N+1. \end{cases}$$

Here we get

$$\frac{\hat{h}_\ell}{\hat{h}_{\ell+1}} = \begin{cases} \frac{1}{q} & \text{for } \ell = 1, \ldots, N-1, \\ 1 & \text{for } \ell = N. \end{cases}$$

The mesh condition (2.20) then reads

$$\lambda_0 = 4 - \frac{1}{q} - q > 0$$

implying $q < 2 + \sqrt{3}$.

Next we consider an example for $m = 2$, in particular a triangulation of the unit square $\mathcal{T} = (0,1)^2$ as constructed in [28, Example 2.1], see Figure 2.1. The coarse mesh ($L = 0$) consists of 16 uniform triangles, at any refinement step each triangle having a vertex on the left edge of $\mathcal{T}$ is cut into four new triangles by connecting the midpoints of its edges. To avoid hanging nodes, additional triangles are refined following the rules as given in [28]. In [28] it was shown that this mesh violates the condition needed in [34] to ensure the stability of the L_2 projection in $H^1(\mathcal{T})$.

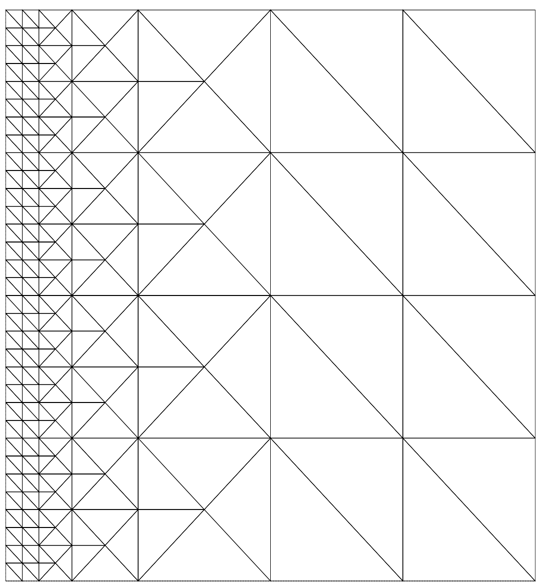


Fig. 2.1. Mesh for $L = 3$ with $M = 168$ nodes

In Table 2.1 we give the computational results to find the constant λ_0 as minimal value over all finite elements τ_ℓ for $s = 1/2$ and $s = 1$, respectively. Here, L is the mesh refinement level, N is the number of finite elements and

M is the number of vertices. We conclude that the L_2 projection on this mesh is stable both in $H^{1/2}(\mathcal{T})$ and $H^1(\mathcal{T})$, respectively.

L	0	1	2	3	4	5	6
M	25	46	87	168	329	650	1291
N	32	68	140	284	572	1148	2300
$s = 1/2$	2.00	1.95	1.92	1.88	1.88	1.88	1.88
$s = 1$	2.00	1.82	1.65	1.49	1.49	1.49	1.49

Table 2.1. Computational results for λ_0.

Remark 2.8. In [26] it is shown, that a triangulation into right isosceles triangles as considered in the previous example always satisfies the mesh condition (2.20). This is due to the definition of $\hat{h}_k$ by the minimal distance of associated vertices, see (1.56).

Finally we consider an adaptive finite element mesh as shown in Figure 2.1 generated by an adaptive algorithm as described in [61]. In Table 2.2 we give values for λ_0 as a function of the refinement level L and the number of finite element nodes M.

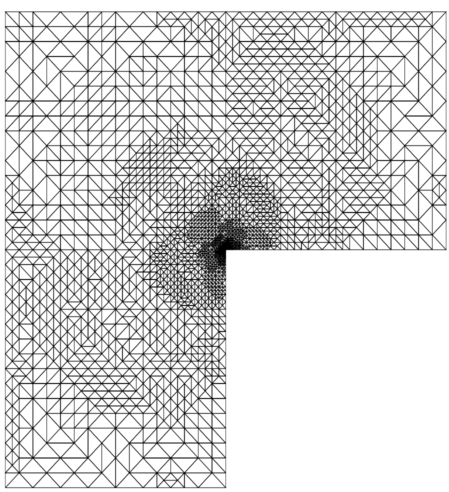


Fig. 2.2. Adaptive finite element triangulation.

L	0	1	2	3	4	5	6	7	8	9
M	8	17	28	53	87	155	291	532	1034	2003
λ_0	2.00	1.93	1.59	1.52	1.52	1.66	1.61	1.09	1.49	1.50

Table 2.2. Computational results for λ_0, adaptive refinement, $s = 1$.

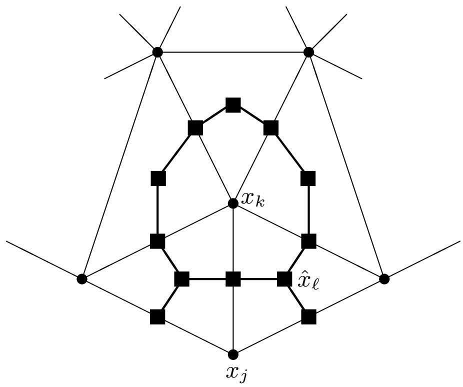


Fig. 2.3. The dual finite element $\widetilde{\tau}_k$.

2.2 Dual Finite Element Spaces

In this section we consider a generalized L_2 projection $\widetilde{Q}_h$ defined by the Galerkin–Petrov variational problem (1.67). Here we restrict our considerations to the case $m = 1$ or $m = 2$. Let $V_h \subset H^1(\mathcal{T})$ be the finite element space spanned by piecewise linear nodal basis functions φ_k, $k = 1, \ldots, M$, see (2.15). To define the test space W_h we use piecewise constant basis functions which are defined with respect to a dual mesh of $\mathcal{T}_h$. To construct such a dual mesh $\widetilde{\mathcal{T}}_M$ we proceed as it is usually done in finite volume methods, see for example [35]. The dual finite element $\widetilde{\tau}_k$ associated with an interior vertex x_k of $\mathcal{T}_N$ is defined by the midpoints $\hat{x}_\ell$ of τ_ℓ. For $m = 2$ we add the midpoints of the related element edges to define $\widetilde{\tau}_k$, see Figure 2.3. Note, that one has to apply appropriate modifications when x_j is a boundary node. Now,

$$\widetilde{\mathcal{T}}_M := \bigcup_{k=1}^{M} \widetilde{\tau}_k \tag{2.23}$$

is the dual mesh of $\mathcal{T}_N$. With respect to $\widetilde{\mathcal{T}}_M$ we define the trial space

$$W_h = \operatorname{span}\{\psi_k\}_{k=1}^{M} \subset L_2(\mathcal{T}) \tag{2.24}$$

of piecewise constant basis functions ψ_k with support $\widetilde{\tau}_k$ for $k = 1, \ldots, M$.

The stiffness matrix of the generalized L_2 projection $\widetilde{Q}_h$ defined by (1.67) is given by

$$\widetilde{G}_h[j,k] \;=\; \langle \varphi_k, \psi_j \rangle_{L_2(\mathcal{T})} \quad \text{for } k,\ell = 1, M. \tag{2.25}$$

For each element τ_ℓ and using the finite element spaces V_h and W_h as defined in (2.15) and (2.24) we compute the local stiffness matrices $\widetilde{G}_\ell$, see (1.61).

For $m = 1$ we get, see (1.60) and (1.61),

$$\hat{G}_\ell \;=\; \frac{1}{2} \cdot \Delta_\ell \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \widetilde{G}_\ell \;=\; \frac{1}{8} \cdot \Delta_\ell \cdot \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

and therefore

$$\frac{3}{4} \cdot (D_\ell \underline{x}, \underline{x}) \;\le\; (\widetilde{G}_\ell \underline{x}, \underline{x}) \;\le\; \frac{3}{2} \cdot (D_\ell \underline{x}_2, \underline{x}) \quad \text{for all } \underline{x} \in \mathbb{R}^2 \tag{2.26}$$

as well as

$$\frac{3}{2} \cdot (D_\ell \underline{x}, \underline{x}) \;=\; (\widetilde{G}_\ell \underline{x}, \underline{x}) \quad \text{for all } \underline{x} \in \mathbb{R}^2. \tag{2.27}$$

Therefore, Assumption (1.2) is satisfied. To check Assumption 2.1 we compute, as in (2.7),

$$\widetilde{G}_\ell^S \;=\; \frac{1}{16} \cdot \Delta_\ell \cdot A_\ell \quad \text{with } A_\ell := \begin{pmatrix} 6 & \dfrac{\hat{h}_1^s}{\hat{h}_2^s} + \dfrac{\hat{h}_2^s}{\hat{h}_1^s} \\ \dfrac{\hat{h}_1^s}{\hat{h}_2^s} + \dfrac{\hat{h}_2^s}{\hat{h}_1^s} & 6 \end{pmatrix}.$$

Hence we have to compute the eigenvalues of A_ℓ, which are given by

$$\lambda_{1/2} \;=\; 6 \pm \left(\frac{\hat{h}_1^s}{\hat{h}_2^s} + \frac{\hat{h}_2^s}{\hat{h}_1^s} \right) \qquad \text{for } m = 1.$$

For $m = 2$ we define the dual mesh locally via a reference element as shown in Figure 2.4. Then we get

$$\hat{G}_\ell \;=\; \frac{1}{3} \cdot \Delta_\ell \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{G}_\ell \;=\; \frac{1}{108} \cdot \Delta_\ell \cdot \begin{pmatrix} 22 & 7 & 7 \\ 7 & 22 & 7 \\ 7 & 7 & 22 \end{pmatrix}$$

and therefore

$$\frac{5}{6} \cdot (D_\ell \underline{x}, \underline{x}) \;\le\; (\widetilde{G}_\ell \underline{x}, \underline{x}) \;\le\; 2 \cdot (D_\ell \underline{x}, \underline{x}) \quad \text{for all } \underline{x} \in \mathbb{R}^3. \tag{2.28}$$

as well as

$$2 \cdot (D_\ell \underline{x}, \underline{x}) \;=\; (\hat{G}_\ell \underline{x}, \underline{x}) \quad \text{for } \underline{x} \in \mathbb{R}^3. \tag{2.29}$$

Therefore, Assumption 1.2 is satisfied. Moreover,

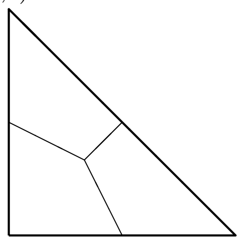


Fig. 2.4. Reference element for $m = 2$.

$$\widetilde{G}_\ell^S = \frac{7}{216} \cdot \Delta_\ell \cdot A_\ell \quad \text{with } A_\ell := \begin{pmatrix} \frac{44}{7} & \frac{\hat{h}_1^s}{\hat{h}_2^s} + \frac{\hat{h}_2^s}{\hat{h}_1^s} & \frac{\hat{h}_1^s}{\hat{h}_3^s} + \frac{\hat{h}_3^s}{\hat{h}_1^s} \\ \frac{\hat{h}_2^s}{\hat{h}_1^s} + \frac{\hat{h}_1^s}{\hat{h}_2^s} & \frac{44}{7} & \frac{\hat{h}_2^s}{\hat{h}_3^s} + \frac{\hat{h}_3^s}{\hat{h}_2^s} \\ \frac{\hat{h}_3^s}{\hat{h}_1^s} + \frac{\hat{h}_1^s}{\hat{h}_3^s} & \frac{\hat{h}_3^s}{\hat{h}_2^s} + \frac{\hat{h}_2^s}{\hat{h}_3^s} & \frac{44}{7} \end{pmatrix}.$$

The eigenvalues of A_ℓ are

$$\lambda_1 = \frac{30}{7}, \qquad \lambda_{2,3} = \frac{51}{7} \pm \sqrt{\sum_{i=1}^{3} \hat{h}_i^{2s} \sum_{i=1}^{3} \frac{1}{\hat{h}_i^{2s}}} \quad \text{for } m = 2.$$

To summarize the results of this section, the stability assumption (2.6) is satisfied if

$$\alpha_m - \sqrt{\sum_{k \in J(\ell)} \hat{h}_k^{2s} \sum_{k \in J(\ell)} \hat{h}_k^{-2s}} \geq c_0 > 0 \tag{2.30}$$

holds for all finite elements τ_ℓ with a global positive constant c_0 where

$$\alpha_m = \begin{cases} 6 & \text{for } m = 1, \\ 51/7 & \text{for } m = 2. \end{cases}$$

Note that the local mesh conditions (2.30) are weaker than (2.20) when using piecewise linear test and trial functions.

2.3 Higher Order Finite Element Spaces

In this section we consider the case when using finite element spaces of higher order polynomials to define the L_2 projection Q_h by a Galerkin–Bubnov approach. Without loss of generality we restrict our considerations to the case $s = 1$. Due to the complicate structure of the local stiffness matrices G_ℓ it is in general impossible to get explicit conditions as in the case of piecewise linear basis functions, see (2.20). Instead, one can approximate the minimal eigenvalue of the symmetrized scaled mass matrix G_ℓ^S numerically. Using a geometrically refined mesh as defined in (2.21), we investigate the mesh condition (2.6) for different trial spaces. Here we restrict our considerations to the case $m = 1$, however, this approach can be applied for an arbitraryly given mesh for $m = 2$ or $m = 3$.

Let τ_ℓ be an arbitrary finite element of the geometrically refined mesh (2.21) defined by its vertices x_ℓ and $x_{\ell+1}$. Then, $h_\ell = |x_{\ell+1} - x_\ell|$. We will use the parametrisation

$$\tau_\ell := \left\{ x(s) \;=\; x_\ell + \frac{1}{2}(s+1)\cdot(x_{\ell+1} - x_\ell) \quad \text{for } s \in [-1,1] \right\} .$$

Let $p \in \mathbb{N}$ be a given polynomial degree which may be different on several finite elements τ_ℓ. To define the local trial space $V_h(\tau_\ell)$ we use form functions $\varphi_\ell^j(x) = \varphi^j(x(s))$ which are defined on the parameter interval $[-1,1]$,

$$V_h(\tau_\ell) \;:=\; \operatorname{span}\{\varphi_\ell^j\}_{j=0}^p\,, \quad \dim V_h(\tau_\ell) \;=\; p+1.$$

The local Galerkin stiffness matrix G_ℓ, which is symmetric, is now given by

$$G_\ell^p[j,k] = \int\limits_{\tau_\ell} \varphi_\ell^k(x(s))\varphi_\ell^j(x(s))dx \;=\; \frac{1}{2}h_\ell \int\limits_{-1}^{1} \varphi_p^k(s)\varphi_p^j(s)ds.$$

Computing the eigenvalues of $D_\ell^{-1}G_\ell$ we can check Assumption 1.1 while computing the eigenvalues of

$$G_\ell^S \;:=\; \frac{1}{2}\left[H_\ell G_\ell H_\ell^{-1} + H_\ell^{-1} G_\ell H_\ell\right]$$

we can investigate Assumption 2.1.

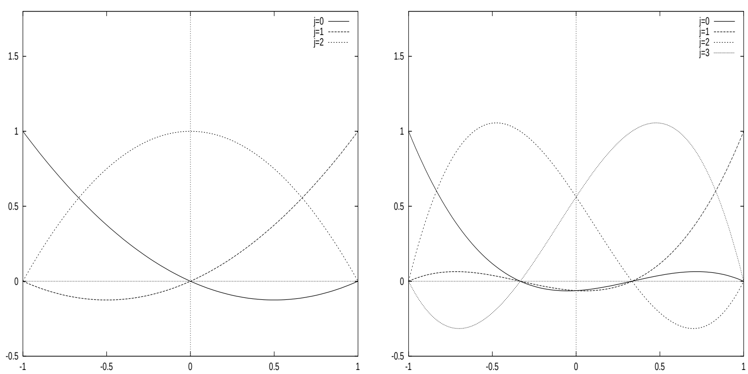
Lagrange Polynomials

Let us first consider the case when using Lagrange polynomials locally to define the trial space $V_h(\tau_\ell)$. For piecewise linear basis functions ($p = 1$) this was already considered in Section 2.1. For $p = 2$ the form functions are given by

$$\varphi_2^0(s) = \frac{1}{2}(s^2 - s), \quad \varphi_2^1(s) = \frac{1}{2}(s^2 + s), \quad \varphi_2^2(s) = 1 - s^2.$$

For $p = 3$ we have

$$\begin{aligned}
\varphi_3^0(s) &= \frac{1}{16}(-9s^3 + 9s^2 + s - 1),\\
\varphi_3^1(s) &= \frac{1}{16}(9s^3 + 9s^2 - s - 1),\\
\varphi_3^2(s) &= \frac{1}{16}(27s^3 - 9s^2 - 27s - 9),\\
\varphi_3^3(s) &= \frac{1}{16}(-27s^3 - 9s^2 + 27s + 9).
\end{aligned}$$



$p = 2$ $p = 3$

Fig. 2.5. Lagrange polynomials.

In [34, Theorem 2] it was shown that the L_2 projection Q_h is stable in $H^1([0,1])$ if

$$\frac{h_i}{h_j} \leq C_0 \cdot \alpha^{|i-j|}, \quad \alpha \leq (1+p)^2 \tag{2.31}$$

is satisfied. When using the geometrically refined mesh (2.21), this gives as upper bound for q,

$$q \leq C_0 \cdot (1+p)^2 . \tag{2.32}$$

Now we will apply the theory developed here to investigate the stability of the associated L_2 projection Q_h. For this we have to compute the local mass matrix G_ℓ^p, we obtain

$$G_\ell^2 = \frac{h_\ell}{30} \cdot \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{pmatrix}, \quad G_\ell^3 = \frac{h_\ell}{1680} \begin{pmatrix} 128 & 19 & 99 & -36 \\ 19 & 128 & -36 & 99 \\ 99 & -36 & 648 & -81 \\ -36 & 99 & -81 & 648 \end{pmatrix}.$$

Let $D_\ell^p = \operatorname{diag} G_\ell^p$. Then we compute the extremal eigenvalues of $(D_\ell^p)^{-1} G_\ell^p$ to get

$$\lambda_{\min}\left((D_\ell^2)^{-1} G_\ell^2\right) = \frac{1}{2}, \quad \lambda_{\max}\left((D_\ell^2)^{-1} G_\ell^2\right) = \frac{5}{4},$$
$$\lambda_{\min}\left((D_\ell^3)^{-1} G_\ell^3\right) = \frac{1}{2}, \quad \lambda_{\max}\left((D_\ell^3)^{-1} G_\ell^3\right) = \frac{189}{128}.$$

Therefore, Assumption 1.1 is satisfied. To investigate Assumption 2.1 we compute the minimal eigenvalue for the scaled matrix $(D_\ell^p)^{-1} G_\ell^p$. The diagonal matrix H_ℓ is hereby defined by using (2.22). For $p = 1, 2, 3$ we compute the constant c_0 in (2.6) for different values of q. The results are given in Figure 2.6 and confirm the theoretical estimate (2.32). In particular, for increasing polynomial degrees, a stronger geometrically refinement is allowed.

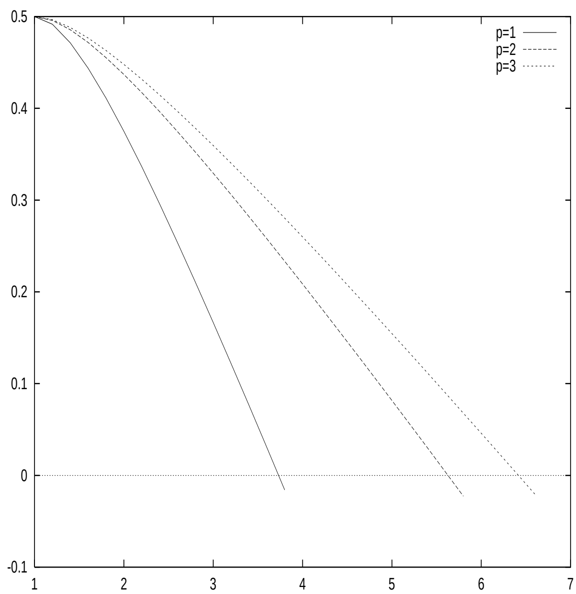


Fig. 2.6. Values of c_0 in (2.6) for varying q.

Antiderivatives of Legendre Polynomials

Instead of Lagrange polynomials we now use antiderivatives of Legendre polynomials to define the form functions $\varphi^j(s)$, $s \in [-1,1]$,

$$\varphi^0(s) := \frac{1}{2}(1-s), \quad \varphi^1(s) := \frac{1}{2}(1+s),$$
$$\varphi^j(s) := -2^{j-1} \int_{-1}^{s} L_{j-1}(t)dt \quad \text{for } j = 2, \ldots, p.$$

Note that the piecewise linear form functions φ^0 and φ^1 define some global basis functions while the higher order form functions are locally. Moreover, the trial space $V_h(\tau_\ell)$ is now defined hierarchically. As an example we consider the form functions for $p = 4$, see Figure 2.7:

$$\varphi^2(s) = -s^2 + 1,$$
$$\varphi^3(s) = -2s^3 + 2s,$$
$$\varphi^4(s) = -5s^4 + 6s^2 - 1.$$

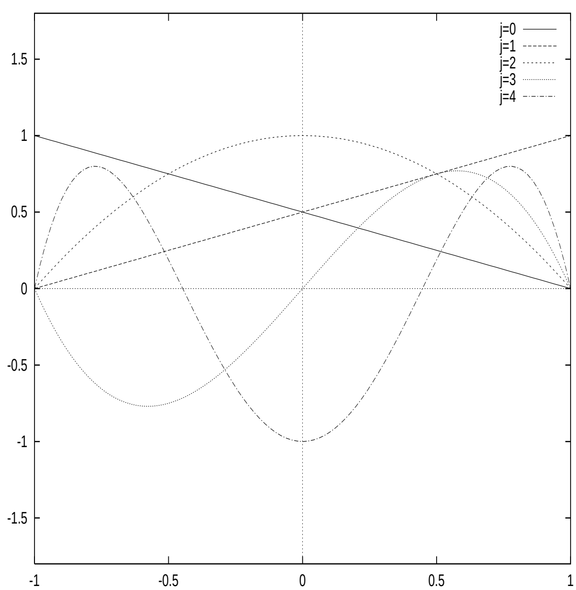


Fig. 2.7. Antiderivatives of Legendre polynomials, $p = 4$.

For $p = 4$ we compute the local mass matrix to obtain

$$G_\ell^4 = \frac{1}{630} \cdot h_\ell \cdot \begin{pmatrix} 210 & 105 & 210 & -84 & 0 \\ 105 & 210 & 210 & 84 & 0 \\ 210 & 210 & 336 & 0 & -96 \\ -84 & 84 & 0 & 192 & 0 \\ 0 & 0 & -96 & 0 & 256 \end{pmatrix}.$$

Note that for $p < 4$ the Galerkin matrices G_ℓ^p are given by the related block matrices in G_ℓ^4. Let $D_\ell^p = \operatorname{diag} G_\ell^p$. To investigate Assumption 1.1 we compute the extremal eigenvalues of $(D_\ell^p)^{-1}G_\ell^p$:

p	$\lambda_{\min}((D_\ell^p)^{-1}G_\ell^p)$	$\lambda_{\max}((D_\ell^p)^{-1}G_\ell^p)$
1	0.5000	1.5000
2	0.1044	2.3956
3	0.1044	2.3956
4	0.0354	2.4256

Table 2.3. Numerical results for Assumption 1.1.

Therefore, Assumption 1.1 is satisfied. To investigate Assumption 2.1 for the geometrically refined mesh (2.21) we compute the minimal eigenvalue of $(D_\ell^p)^{-1}G_\ell^p$ for varying q, see Figure 2.8.

Now, the mesh parameter q has to be decreased when the polynomial degree p is increased.

2.4 Biorthogonal Basis Functions

The stability assumption (2.1),

$$(H_\ell \widetilde{G}_\ell^\top H_\ell^{-1} \underline{x}_\ell, \underline{x}_\ell) \geq c_0 \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \quad \text{for all } \underline{x}_\ell \in \mathbb{R}^{M_\ell},$$

is trivially satisfied when $G_\ell^\top$ is a diagonal matrix. Then, $c_0 = 1$. Recall that the local mass matrix $\widetilde{G}_\ell$ is defined by

$$\widetilde{G}_\ell[j,i] = \langle \varphi_i^\ell, \psi_j^\ell \rangle_{L_2(\tau_\ell)} \quad \text{for } i,j = 1, \ldots, M_\ell.$$

For a given trial space $V_h(\tau_\ell)$ we therefore have to find a biorthogonal basis of $W_h(\tau_\ell)$ such that

$$\langle \varphi_i^\ell, \psi_j^\ell \rangle_{L_2(\tau_\ell)} = 0 \quad \text{for } i \neq j.$$

Biorthogonal basis functions are well established in the context of wavelet approximations of boundary integral equations, see [56] and the references given there. In [74, 75] they were adapted to Mortar finite element methods.

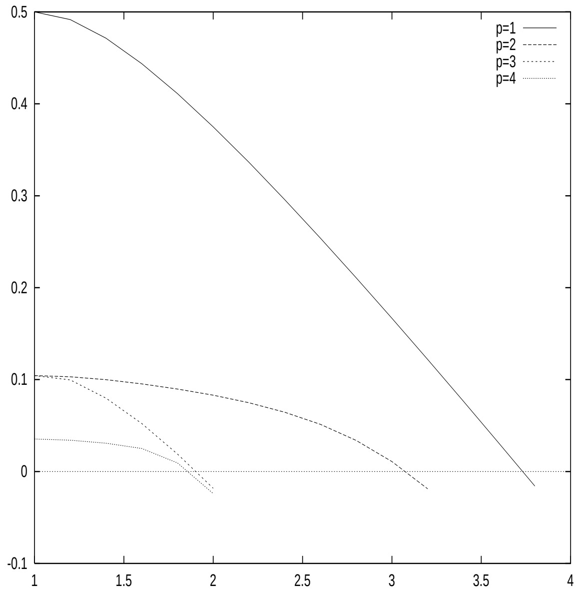


Fig. 2.8. Values of c_0 in (2.6) for varying q.

Here we will recall the definition of biorthogonal basis functions for $m = 1$ to illustrate how this approach fits into the general theory presented here. Let $V_h(\tau_\ell)$ the local trial space of piecewise linear form functions, see (1.45). Let the finite element τ_ℓ be parametrized as

$$\tau_\ell := \{x(s) = x_\ell + s \cdot (x_{\ell+1} - x_\ell) \quad \text{for } s \in [0,1]\} .$$

To define $V_h(\tau_\ell)$ we have $\varphi_j^\ell(x(s)) = \varphi_j(s)$ using form functions given as

$$\varphi_1(s) = 1 - s, \quad \varphi_2(s) = s \quad \text{for } s \in [0,1].$$

Now we have to find form functions ψ_j satisfying

$$\int_0^s \varphi_i(s)\psi_j(s)ds = \frac{1}{2} \cdot \delta_{ij} \quad \text{for } i,j = 1,2.$$

As in [74] we construct ψ_j either as a piecewise constant or a piecewise linear basis function, see Figure 2.9. For a piecewise constant basis function, a simple computation gives

$$\psi_1^0(s) = \begin{cases} \frac{3}{2} & \text{for } s \in [0, \frac{1}{2}), \\ -\frac{1}{2} & \text{for } s \in [\frac{1}{2}, 1], \end{cases} \tag{2.33}$$

$$\psi_2^0(s) = \begin{cases} -\frac{1}{2} & \text{for } s \in [0, \frac{1}{2}), \\ \frac{3}{2} & \text{for } s \in [\frac{1}{2}, 1]. \end{cases} \tag{2.34}$$

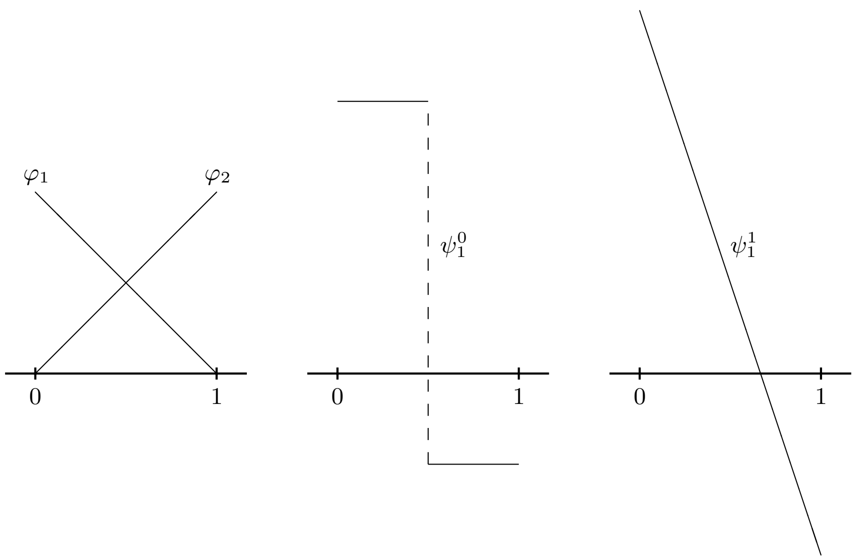


Fig. 2.9. Biorthogonal basis functions.

Now we have to check Assumption 1.2. Since $\widetilde{G}_\ell$ is diagonal, (1.62) is trivially satisfied. To show (1.63) we compute $\hat{G}_\ell$ to obtain

$$\hat{G}_\ell = \frac{h_\ell}{4} \cdot \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}.$$

Then we get

$$\frac{3}{2} \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \le (\hat{G}_\ell \underline{x}_\ell, \underline{x}_\ell) \le 6 \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \quad \text{for all } \underline{x}_\ell \in \mathbb{R}^{M_\ell}$$

and (1.63) is satisfied. For a piecewise linear trial function we obtain the form functions

$$\psi_1^1(s) = 2 - 3s, \quad \psi_2^2(s) = 3s - 1 \quad \text{for } s \in [0, 1]. \tag{2.35}$$

Then,

$$\hat{G}_\ell = \frac{h_\ell}{2} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and therefore

$$3 \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \le (\hat{G}_\ell \underline{x}_\ell, \underline{x}_\ell) \le 9 \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \quad \text{for all } \underline{x}_\ell \in \mathbb{R}^{M_\ell}.$$

Hence, Assumption 1.2 is satisfied.

For higher order polynomial basis functions as well as for $m = 2$ we can consider corresponding biorthogonal basis functions in a similar way, see [74].

3

The Dirichlet–Neumann Map for Elliptic Boundary Value Problems

By solving an elliptic second order boundary value problem with Dirichlet boundary conditions the associated Neumann data are well defined. This Dirichlet–Neumann map can be used to solve boundary value problems with mixed boundary conditions (see Chapter 4) as well as in domain decomposition methods (see Chapter 5). The Dirichlet–Neumann map can be written as

$$\gamma_1 u(x) \,=\, S\gamma_0 u(x) - Nf(x) \quad \text{for } x \in \Gamma.$$

We will use a domain variational formulation (see (3.16)) as well as boundary integral equations (see (3.40) and (3.55)) to describe and to analyze the Steklov–Poincaré operator S and the Newton potential Nf. Since both representations are given implicitly, we have to define suitable approximations to be used in practical computations. This is done by using finite and boundary element methods leading to approximations having similar analytic and algebraic properties.

The Dirichlet–Neumann map was originally introduced in [2]; see also [55] for a finite element approach and [44] for a coupled finite and boundary element approach.

Let $\Omega \subset \mathbb{R}^n$ and $n = 2$ or $n = 3$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$ which is decomposed into non–overlapping parts Γ_D and Γ_N. We assume $\text{meas}_{n-1}\,\Gamma_D > 0$. As a model problem we consider a scalar second order uniformly elliptic boundary value problem with mixed boundary conditions of Dirichlet and Neumann type, respectively:

$$\begin{aligned} L(x)u(x) &= f(x) && \text{for } x \in \Omega, \\ \gamma_0 u(x) &= g_D(x) && \text{for } x \in \Gamma_D, \\ \gamma_1 u(x) &= g_N(x) && \text{for } x \in \Gamma_N. \end{aligned} \tag{3.1}$$

Instead of a scalar problem we may consider systems as well, then all further assumptions have to be formulated in an appropriate way. We assume

O. Steinbach: LNM 1809, pp. 53–70, 2003.

$f \in \widetilde{H}^{-1}(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$ and $g_N \in H^{-1/2}(\Gamma_N)$. In (3.1), the partial differential operator $L(x)$, $x \in \Omega$, is given by

$$L(x)u(x) \;=\; -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left[a_{ji}(x)\frac{\partial}{\partial x_i}u(x)\right] \tag{3.2}$$

with $a_{ji} = a_{ij} \in L_\infty(\Omega)$, $i,j = 1,\ldots,n$. $L(\cdot)$ is assumed to be uniformly elliptic, in particular, there exists a positive constant c_0 independent of $x \in \Omega$ such that for all $x \in \Omega$,

$$\sum_{k,\ell=1}^{n} a_{k\ell}(x)\xi_k\xi_\ell \;\geq\; c_0 \cdot |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{3.3}$$

In addition, $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ is the trace operator and the associated conormal derivative operator γ_1 is given by

$$\gamma_1 u(x) \;:=\; \sum_{i,j=1}^{n} n_j(x)a_{ji}(x)\frac{\partial}{\partial x_i}u(x) \quad \text{for } x \in \Gamma \tag{3.4}$$

where $n(x)$ is the exterior unit normal vector defined almost everywhere for $x \in \Gamma$. For $u, v \in H^1(\Omega)$ we define the symmetric bilinear form

$$a(u,v) \;:=\; \sum_{i,j=1}^{n} \int_\Omega \frac{\partial}{\partial x_j}v(x)a_{ji}(x)\frac{\partial}{\partial x_i}u(x)dx$$

which is bounded in $H^1(\Omega)$,

$$|a(u,v)| \;\leq\; c_2^A \cdot ||u||_{H^1(\Omega)}||v||_{H^1(\Omega)} \quad \text{for all } u,v \in H^1(\Omega). \tag{3.5}$$

Now we can write Green's second formula for $u, v \in H^1(\Omega)$,

$$a(u,v) \;=\; \int_\Omega Lu(x)v(x)dx + \int_\Gamma \gamma_1 u(x)\gamma_0 v(x)ds_x. \tag{3.6}$$

Using (3.6), the variational formulation of the mixed boundary value problem (3.1) is: find $u \in H^1(\Omega)$ satisfying $\gamma_0 u(x) = g(x)$ for $x \in \Gamma_D$ such that

$$a(u,v) \;=\; \int_\Omega f(x)v(x)dx + \int_{\Gamma_N} g_N(x)\gamma_0 v(x)ds_x \tag{3.7}$$

for all $v \in H^1(\Omega)$ with $\gamma_0 v(x) = 0$ for $x \in \Gamma_D$.

Let

$$H_0^1(\Omega,\Gamma_D) \;:=\; \left\{v \in H^1(\Omega) \,:\, \gamma_0 v(x) = 0 \quad \text{for } x \in \Gamma_D\right\}. \tag{3.8}$$

From the uniform ellipticity (3.3) and using the Poincaré inequality we get that the bilinear form $a(\cdot,\cdot)$ is elliptic on $H^1_0(\Omega,\Gamma_D)$,

$$a(v,v) \;\geq\; c_1^A \cdot ||v||^2_{H^1(\Omega)} \quad \text{for all } v \in H^1_0(\Omega,\Gamma_D). \tag{3.9}$$

Now, using (3.5) and (3.9) we can apply the Lax–Milgram theorem to prove the following result, see for example [52, Theorem 4.10]:

Theorem 3.1. *Let the bilinear form $a(\cdot,\cdot)$ be bounded and elliptic on the space $H^1_0(\Omega,\Gamma_D)$. Then the variational problem (3.7) has a unique solution $u \in H^1(\Omega)$ satisfying*

$$||u||_{H^1(\Omega)} \;\leq\; c \cdot \Big\{ ||f||_{\widetilde{H}^{-1}(\Omega)} + ||g_D||_{H^{1/2}(\Gamma_D)} + ||g_N||_{H^{-1/2}(\Gamma_N)} \Big\}. \tag{3.10}$$

Instead of the mixed boundary value problem (3.1) we also consider the Dirichlet boundary value problem

$$L(x)u(x) \;=\; f(x) \quad \text{for } x \in \Omega, \qquad \gamma_0 u(x) = g(x) \quad \text{for } x \in \Gamma \tag{3.11}$$

to define an associated Dirichlet–Neumann map. Applying Theorem 3.1 there exists a unique weak solution $u \in H^1(\Omega)$ of (3.11). Hence, by (3.4) we can compute the conormal derivative $\lambda := \gamma_1 u$. In what follows we prove that $\lambda \in H^{-1/2}(\Gamma)$, see also [52, Lemma 4.3]. For $u \in H^1(\Omega)$ being the unique solution of (3.11) we define the linear functional

$$\ell(w) \;:=\; a(u,\mathcal{E}w) - \int_\Omega f(x)\mathcal{E}w(x)dx \quad \text{for } w \in H^{1/2}(\Gamma). \tag{3.12}$$

where $\mathcal{E} : H^{1/2}(\Gamma) \to H^1(\Omega)$ is a bounded extension operator, see Theorem 1.1. Using (3.10) we get

$$|\ell(w)| \;\leq\; c \cdot \Big\{ ||g||_{H^{1/2}(\Gamma)} + ||f||_{\widetilde{H}^{-1}(\Omega)} \Big\} \cdot ||w||_{H^{1/2}(\Gamma)} \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.13}$$

Applying the Riesz representation theorem, there exists a $\lambda \in H^{-1/2}(\Gamma)$ such that

$$\langle \lambda, w \rangle_{L_2(\Gamma)} \;=\; \ell(w) \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.14}$$

Hence, the conormal derivative $\lambda \in H^{-1/2}(\Gamma)$ satisfies

$$\int_\Gamma \lambda(x)w(x)ds_x \;=\; a(u_0 + \mathcal{E}g, \mathcal{E}w) - \int_\Omega f(x)\mathcal{E}w(x)dx \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.15}$$

By doing so, we have defined a map from the given data (f, g) to the associated Neumann boundary data $\lambda := \gamma_1 u$. In particular, for fixed f and varying Dirichlet boundary data $g = \gamma_0 u$ we have defined a Dirichlet–Neumann map which we may write as

$$\gamma_1 u(x) \; = \; Sg(x) - Nf(x) \quad \text{for } x \in \Gamma. \tag{3.16}$$

Here, S is the Steklov–Poincaré operator and Nf is some Newton potential. The mixed boundary value problem (3.1) is then equivalent to find the Dirichlet data u with $u(x) = g(x)$ for $x \in \Gamma_D$ such that

$$g_N(x) \; = \; \gamma_1 u(x) \; = \; Su(x) - Nf(x) \qquad \text{for } x \in \Gamma_N. \tag{3.17}$$

In Chapter 4 we will investigate the boundary integral equation (3.17) to ensure unique solvability. Then we describe different discretization techniques to solve (3.17) numerically.

3.1 The Steklov–Poincaré Operator

To define and to analyze the Steklov–Poincaré operator S used in (3.16), we first consider the homogeneous Dirichlet boundary value problem

$$L(x)u(x) \; = \; 0 \quad \text{for } x \in \Omega, \quad \gamma_0 u(x) = g(x) \quad \text{for } x \in \Gamma \tag{3.18}$$

where $g \in H^{1/2}(\Gamma)$ is given. The variational problem is: find $u \in H^1(\Omega)$ with $\gamma_0 u(x) = g(x)$ for $x \in \Gamma$ such that

$$a(u, v) \; = \; 0 \quad \text{for all } v \in H^1_0(\Omega, \Gamma). \tag{3.19}$$

Due to Theorem 3.1 there exists a unique solution $u \in H^1(\Omega)$ of (3.19) and we can compute its conormal derivative $\lambda(x) := \gamma_1 u(x)$ for $x \in \Gamma$ almost everywhere. Using (3.6) the related variational problem is: find $\lambda \in H^{-1/2}(\Gamma)$ such that

$$\int_\Gamma \lambda(x) w(x) ds_x \; = \; a(u, \mathcal{E}w) \quad \text{for all } w \in H^{1/2}(\Gamma) \tag{3.20}$$

where $\mathcal{E}w \in H^1(\Omega)$ is a bounded extension of $w \in H^{1/2}(\Gamma)$, see Theorem 1.1.

Let $\widetilde{g} := \mathcal{E}g \in H^1(\Omega)$ be a bounded extension of the given Dirichlet datum $g \in H^{1/2}(\Gamma)$ and let us define the bounded bilinear form

$$b(w, \mu) \; := \; \int_\Gamma w(x)\mu(x) ds_x \; : \; H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}. \tag{3.21}$$

Then we may write (3.19) and (3.20) as a coupled variational problem to find $(u_0, \lambda) \in H^1_0(\Omega, \Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} a(u_0 + \widetilde{g}, v) &= 0 \\ b(w, \lambda) \qquad &= a(u_0 + \widetilde{g}, \mathcal{E}w) \end{aligned} \tag{3.22}$$

for all $(v, w) \in H^1_0(\Omega, \Gamma) \times H^{1/2}(\Gamma)$.

Theorem 3.2. *For any $g \in H^{1/2}(\Gamma)$ there exists a unique solution $\lambda \in H^{-1/2}(\Gamma)$ of the coupled variational problem (3.22) satisfying*

$$||\lambda||_{H^{-1/2}(\Gamma)} \leq c \cdot ||g||_{H^{1/2}(\Gamma)}. \tag{3.23}$$

Proof. Applying Theorem 3.1 we first have that there exists a unique solution $u_0 \in H^1_0(\Omega, \Gamma)$ satisfying

$$a(u_0 + \widetilde{g}, v) = 0 \quad \text{for all } v \in H^1_0(\Omega, \Gamma)$$

and

$$||u_0||_{H^1(\Omega)} \leq \frac{c_2^A}{c_1^A} \cdot ||\widetilde{g}||_{H^1(\Omega)}.$$

Defining $u := u_0 + \widetilde{g} \in H^1(\Omega)$ we obtain, by applying Theorem 1.1,

$$||u||_{H^1(\Omega)} \leq c_T \cdot (1 + \frac{c_2^A}{c_1^A}) \cdot ||g||_{H^{1/2}(\Gamma)}.$$

It remains to prove the solvability of the variational problem to find $\lambda \in H^{-1/2}(\Gamma)$ such that

$$b(w, \lambda) = \langle w, \lambda \rangle_{L_2(\Gamma)} = a(u, \mathcal{E}w) \quad \text{for all } w \in H^{1/2}(\Gamma).$$

By setting $X := H^{1/2}(\Gamma)$, $\Pi := H^{-1/2}(\Gamma)$ and $B = I$ this corresponds to the general situation as described in Theorem 1.2. Hence we have to check the inf–sup condition (1.18). Since $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$ with respect to the L_2 inner product, we therefore have

$$||\mu||_{H^{-1/2}(\Gamma)} = \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{|\langle w, \mu \rangle_{L_2(\Gamma)}|}{||w||_{H^{1/2}(\Gamma)}} \quad \text{for all } \mu \in H^{-1/2}(\Gamma)$$

implying the inf–sup condition (1.18) with $\gamma_S = 1$. Using Theorem 1.2 this gives unique solvability of the second variational problem in (3.22). Moreover, with

$$||\lambda||_{H^{-1/2}(\Gamma)} = \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{|\langle w, \lambda \rangle_{L_2(\Gamma)}|}{||w||_{H^{1/2}(\Gamma)}} = \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{a(u_0 + \widetilde{g}, \mathcal{E}w)}{||w||_{H^{1/2}(\Gamma)}}$$

$$\leq c_{IT} \cdot c_2^A \cdot ||u||_{H^1(\Omega)} \leq c_{IT} \cdot c_2^A \cdot c_T \cdot (1 + \frac{c_2^A}{c_1^A}) \cdot ||g||_{H^{1/2}(\Gamma)}$$

we get the estimate (3.23). □

By solving the coupled variational problem (3.22) we have defined a linear operator mapping some given Dirichlet data $g = \gamma_0 u$ to the associated Neumann data $\lambda = \gamma_1 u$,

$$Sg(x) := \lambda(x) \quad \text{for } x \in \Gamma. \tag{3.24}$$

In particular, by the Riesz representation theorem we have the identity

$$\langle Sg, w\rangle_{L_2(\Gamma)} \;=\; b(w,\lambda) \quad \text{for all } w \in H^{-1/2}(\Gamma). \tag{3.25}$$

From Theorem 3.2 it is obvious that $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a bounded operator with

$$||Sg||_{H^{-1/2}(\Gamma)} \;\leq\; c \cdot ||g||_{H^{1/2}(\Gamma)} \quad \text{for all } g \in H^{1/2}(\Gamma). \tag{3.26}$$

Let us define the function space

$$H_0^{1/2}(\Gamma,\Gamma_D) \;:=\; \Big\{ v \in H^{1/2}(\Gamma) \;:\; \gamma_0 v(x) = 0 \quad \text{for } x \in \Gamma_D \Big\} \tag{3.27}$$

where $\mathrm{meas}_{n-1}\Gamma_D > 0$. Note that

$$H_0^{1/2}(\Gamma,\Gamma_D) \;=\; \widetilde{H}^{1/2}(\Gamma_N)\,.$$

Now we can prove the ellipticity of the Steklov–Poincaré operator S on the subspace $H_0^{1/2}(\Gamma,\Gamma_D)$, see also [2].

Theorem 3.3. *The Steklov–Poincaré operator defined by (3.24) is elliptic on* $H_0^{1/2}(\Gamma,\Gamma_D)$,

$$\langle Sv, v\rangle_{L_2(\Gamma)} \;\geq\; c_1^S \cdot ||v||^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v \in H_0^{1/2}(\Gamma,\Gamma_D). \tag{3.28}$$

Proof. For an arbitrary but fixed $g \in H_0^{1/2}(\Gamma,\Gamma_D)$ we have $g(x) = 0$ for $x \in \Gamma_D$. Then, for a bounded extension $\widetilde{g} := \mathcal{E}g \in H^1(\Omega)$ we clearly have $\gamma_0\widetilde{g} = g(x) = 0$ for $x \in \Gamma_D$ and therefore $\widetilde{g} \in H_0^1(\Omega,\Gamma_D)$. The application of the Steklov–Poincaré operator is then defined by solving the variational problem (3.22). Note that $u_0 \in H_0^1(\Omega,\Gamma)$ and therefore $u := u_0 + \widetilde{g} \in H_0^1(\Omega,\Gamma_D)$. Inserting the definition (3.24) of the Steklov–Poincaré operator S into the second equation of (3.22), we get, adding the first equation of (3.22) with $v = u_0 \in H_0^1(\Omega,\Gamma)$,

$$\begin{aligned}\langle Sg, g\rangle_{L_2(\Gamma)} \;&=\; b(g, Sg) = a(u_0 + \widetilde{g}, \widetilde{g}) \\ &= a(u_0 + \widetilde{g}, u_0 + \widetilde{g}) \;=\; a(u,u) \;\geq\; c_1^A \cdot ||u||^2_{H^1(\Omega)}\end{aligned}$$

by (3.9). Applying the trace theorem (Theorem 1.1) we have

$$||g||_{H^{1/2}(\Gamma)} \;=\; ||\gamma_0 u||_{H^{1/2}(\Gamma)} \;\leq\; c_T \cdot ||u||_{H^1(\Omega)}$$

which completes the proof. □

Up to now we only used a domain variational formulation to define and to describe the Steklov–Poincaré operator S and the Dirichlet–Neumann map (3.16). Since the Dirichlet–Neumann map is in fact a map from some given Dirichlet data on the boundary Γ to some Neumann data on Γ, a description of S by boundary terms only may be favorable in some situations. Hence we will now consider boundary integral operators to define the Steklov–Poincaré operator S.

We assume that there exists a fundamental solution $U^*(\cdot, y)$ of the partial differential operator $L(\cdot)$ in (3.18). This assumption is, for example, satisfied when considering partial differential operators with constant coefficients, see for example the discussion given in [54, p. 31f]. Using a direct approach based on Green's formula, the solution of the homogeneous Dirichlet boundary value problem (3.18) is then given by the representation formula

$$u(x) = \int_\Gamma U^*(x,y)\gamma_1 u(y) ds_y - \int_\Gamma \gamma_1(y) U^*(x,y)\gamma_0 u(y) ds_y \quad \text{for } x \in \Omega. \tag{3.29}$$

Here, $\gamma_1(y)$ denotes the application of the conormal derivative operator with respect to $y \in \Gamma$. To compute the yet unknown Neumann datum $\lambda(x) := \gamma_1 u(x)$ for $x \in \Gamma$ we have to derive a suitable boundary integral equation. By applying the trace operators γ_i $(i = 0, 1)$ to the representation formula (3.29) we obtain a system of boundary integral equations, $x \in \Gamma$,

$$\begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix}. \tag{3.30}$$

Here, the boundary integral operators are defined for $x \in \Gamma$ in the standard way, in particular the single layer potential operator

$$V\lambda(x) = \int_\Gamma U^*(x,y)\lambda(y) ds_y, \tag{3.31}$$

the double layer potential operator

$$Ku(x) = \int_\Gamma \gamma_1(y) U^*(x,y) u(y) ds_y \tag{3.32}$$

and the adjoint double layer potential

$$K'\lambda(x) = \int_\Gamma \gamma_1(x) U^*(x,y)\lambda(y) ds_y \tag{3.33}$$

as well as the hypersingular integral operator

$$Du(x) = -\gamma_1(x) \int_\Gamma \gamma_1(y) U^*(x,y) u(y) ds_y. \tag{3.34}$$

The mapping properties of all boundary integral operators defined above are well known, see e.g. [32, 33]. In particular, the boundary integral operators are bounded for $|s| \leq \frac{1}{2}$:

$$\begin{aligned} V &: H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \\ K &: H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \\ K' &: H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma), \\ D &: H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma). \end{aligned}$$

Moreover, without loss of generality, we assume that the single layer potential V is $H^{-1/2}(\Gamma)$–elliptic satisfying

$$\langle Vw, w\rangle_{L_2(\Gamma)} \geq c_1^V \cdot ||w||^2_{H^{-1/2}(\Gamma)} \quad \text{for all } w \in H^{-1/2}(\Gamma). \tag{3.35}$$

The hypersingular integral operator D is assumed to be $H^{1/2}(\Gamma)$ semi–elliptic,

$$\langle Dw, w\rangle_{L_2(\Gamma)} \geq c_1^D \cdot ||w||^2_{H^{1/2}(\Gamma)} \quad \text{for all } w \in H^{1/2}(\Gamma)_{/\mathbf{R}}. \tag{3.36}$$

Here, $\mathbf{R}$ is the solution space of the homogeneous Neumann boundary value problem $L(x)u(x) = 0$ in Ω and $\gamma_1 u(x) = 0$ on Γ. In fact, the ellipticity inequalities (3.35) and (3.36) follow from the strong ellipticity (3.3) of the underlying partial differential operator $L(\cdot)$. However, for $n = 2$ appropriate scaling conditions are needed to ensure (3.35), see for example [32, 46].

Since the single layer potential V is assumed to be invertible, we get from the first equation in (3.30) the Dirichlet–Neumann map

$$\gamma_1 u(x) = V^{-1}(\frac{1}{2}I + K)\gamma_0 u(x) \quad \text{for } x \in \Gamma. \tag{3.37}$$

Inserting this into the second equation in (3.30), we get an alternative representation of the Dirichlet–Neumann map,

$$\begin{aligned} \gamma_1 u(x) &= D\gamma_0 u(x) + (\frac{1}{2}I + K')\gamma_1 u(x) \\ &= \left[D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\right]\gamma_0 u(x). \end{aligned} \tag{3.38}$$

As in (3.24) we can write the Dirichlet–Neumann map for the homogeneous Dirichlet boundary value problem (3.18) as

$$\gamma_1 u(x) = S\gamma_0 u(x) \quad \text{for } x \in \Gamma$$

using the Steklov–Poincaré operator

$$S\gamma_0 u(x) = V^{-1}(\frac{1}{2}I + K)\gamma_0 u(x) \tag{3.39}$$

$$= \left[D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\right]\gamma_0 u(x). \tag{3.40}$$

Note that more alternative representations of the Steklov–Poincaré operators by boundary integral operators are available, see for example [45]. However, here we will consider the symmetric representation (3.40) only. As it will be seen later, this approach is almost similar to the approach when using a domain variational formulation to define the Steklov–Poincaré operator.

Based on the mapping properties of the boundary integral operators used above we can give alternative proofs of the mapping properties of the Steklov–Poincaré operator S. In particular, $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded, see (3.26). Using the symmetric representation (3.40) we get by using the $H^{-1/2}(\Gamma)$–ellipticity of the single layer potential V the spectral equivalence inequality

$$\langle Sv, v\rangle_{L_2(\Gamma)} \geq \langle Dv, v\rangle_{L_2(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma). \tag{3.41}$$

Hence, using (3.36) we get

$$\langle Sv, v\rangle_{L_2(\Gamma)} \geq c_1^D \cdot ||v||^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma)_{/\mathbf{R}}, \tag{3.42}$$

see Theorem 3.3. Note that using (3.26) and (3.36) we also have

$$\langle Sv, v\rangle_{L_2(\Gamma)} \leq c \cdot \langle Dv, v\rangle_{L_2(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma)_{/\mathbf{R}}. \tag{3.43}$$

Hence, the Steklov–Poincaré operator S is spectrally equivalent to the hypersingular integral operator D. Note that the Steklov–Poincaré operator S is in general given implicitly, while the hypersingular integral operator is given in an explicit form. This becomes important when constructing preconditioners for Galerkin discretizations of the Steklov–Poincaré operator, see for example [27].

3.2 The Newton potential

To describe and to analyze the Newton potential used in the Dirichlet–Neumann map (3.16), we will consider a boundary value problem with homogeneous Dirichlet boundary conditions,

$$L(x)u(x) = f(x) \quad \text{for } x \in \Omega, \quad \gamma_0 u(x) = 0 \quad \text{for } x \in \Gamma. \tag{3.44}$$

Its variational problem is: find $u \in H_0^1(\Omega, \Gamma)$ such that

$$a(u, v) = \int_\Omega f(x)v(x)dx \quad \text{for all } v \in H_0^1(\Omega, \Gamma). \tag{3.45}$$

Due to Theorem 3.1, there exists a unique solution $u \in H_0^1(\Omega, \Gamma)$ of (3.45) satisfying

$$||u||_{H^1(\Omega)} \leq c \cdot ||f||_{\widetilde{H}^{-1}(\Omega)}. \tag{3.46}$$

As before we can compute the associated conormal derivative $\lambda(x) := \gamma_1 u(x)$ for $x \in \Gamma$ by solving the variational problem

$$\int_\Gamma \lambda(x)w(x)ds_x = a(u,\mathcal{E}w) - \int_\Omega f(x)\mathcal{E}w(x)dx \quad \text{for all } w \in H^{1/2}(\Gamma). \quad (3.47)$$

As in Theorem 3.2 we have unique solvability of (3.47):

Theorem 3.4. *For any* $f \in \widetilde{H}^{-1}(\Omega)$ *there exists a unique solution* $\lambda \in H^{-1/2}(\Gamma)$ *satisfying*

$$||\lambda||_{H^{-1/2}(\Gamma)} \le c \cdot ||f||_{\widetilde{H}^{-1}(\Omega)}. \quad (3.48)$$

The proof is essentially based on the inf–sup condition for the bilinear form $b(v,\mu) := \langle v,\mu\rangle_{L_2(\Gamma)}$, see the proof of Theorem 3.2.

Hence, we can define the Newton potential

$$Nf(x) := -\lambda(x) \quad \text{for } x \in \Gamma \quad (3.49)$$

where $\lambda \in H^{-1/2}(\Gamma)$ is the unique solution of (3.47). Now, applying (3.48) we have

$$||Nf||_{H^{-1/2}(\Gamma)} \le c \cdot ||f||_{\widetilde{H}^{-1}(\Omega)} \quad \text{for all } f \in \widetilde{H}^{-1}(\Omega). \quad (3.50)$$

Now, instead of (3.45) and (3.47), we use boundary integral equations to define the Newton potential Nf. For the boundary value problem (3.44) the representation formula is

$$u(x) = \int_\Gamma U^*(x,y)\lambda(y)ds_y + \int_\Omega U^*(x,y)f(y)dy \quad \text{for } x \in \Omega. \quad (3.51)$$

To find the yet unknown Neumann datum $\lambda \in H^{-1/2}(\Gamma)$ we have to solve the boundary integral equation

$$\int_\Gamma U^*(x,y)\lambda(y)ds_y = -\int_\Omega U^*(x,y)f(y)dy \quad \text{for } x \in \Gamma. \quad (3.52)$$

If we define the Newton potential

$$N_0 f(x) := \int_\Omega U^*(x,y)f(y)dy \quad \text{for } x \in \Gamma, \quad (3.53)$$

we get by solving (3.52)

$$\lambda(x) = -V^{-1}N_0 f(x) \quad \text{for } x \in \Gamma. \quad (3.54)$$

Therefore, the Newton potential Nf used in (3.16) is given by

$$Nf(x) := -V^{-1}N_0 f(x) \quad \text{for } x \in \Gamma. \quad (3.55)$$

Note that we may derive (3.50) by using the mapping properties of the single layer potential V and of the Newton potential $N_0 f$.

Since both the Steklov–Poincaré operator S and the Newton potential Nf and therefore the Dirichlet–Neumann map (3.16) are defined only implicitly, namely by solving a variational problem in the domain or on the boundary, we have to define suitable approximations $\widetilde{S}$ and $\widetilde{N}f$ to be used in practical computations. Then, instead of the Dirichlet–Neumann map (3.16) we will consider the modified Dirichlet Neumann map,

$$\widetilde{\lambda}(x) := \widetilde{S}g(x) - \widetilde{N}f(x) \quad \text{for } x \in \Gamma. \tag{3.56}$$

To define these approximations we use either a finite element approximation or a Galerkin boundary element approximation. We will show that both approaches lead to stable approximations with similar properties. In particular, we have to ensure that the approximate Steklov–Poincaré operators $\widetilde{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are bounded, elliptic on $H_0^{1/2}(\Gamma, \Gamma_D)$, and satisfy an approximation property for $||(S - \widetilde{S})v||_{H^{-1/2}(\Gamma)}$. In addition we need some approximation property for the approximate Newton potential, in particular for $||Nf - \widetilde{N}f||_{H^{-1/2}(\Gamma)}$. Then we can make use of the standard theory based on the Strang lemma [29].

3.3 Approximation by Finite Element Methods

In this section we consider a stable finite element approximation $\widetilde{S}$ of the Steklov–Poincaré operator S and a finite element approximation $\widetilde{N}f$ of the Newton potential Nf.

For $g \in H^{1/2}(\Gamma)$ the application Sg of the Steklov–Poincaré operator is given by

$$\langle Sg, w\rangle_{L_2(\Gamma)} = a(u_0 + \widetilde{g}, \mathcal{E}w) \quad \text{for all } w \in H^{1/2}(\Gamma) \tag{3.57}$$

where $u_0 \in H_0^1(\Omega, \Gamma)$ is the unique solution of

$$a(u_0 + \widetilde{g}, v) = 0 \quad \text{for all } v \in H_0^1(\Omega, \Gamma). \tag{3.58}$$

Note that $\widetilde{g} := \mathcal{E}g \in H^1(\Omega)$ is a bounded extension of the given Dirichlet datum $g \in H^{1/2}(\Gamma)$. To define a suitable approximation $\widetilde{S}g$ we introduce a finite dimensional trial space

$$\widetilde{X}_h := \text{span}\,\{\phi_k\}_{k=1}^{\widetilde{M}} \subset H_0^1(\Omega, \Gamma) \tag{3.59}$$

of piecewise polynomial basis functions which are zero on the boundary Γ. The Galerkin approximation of (3.58) is: find $u_{0,h} \in \widetilde{X}_h$ such that

$$a(u_{0,h} + \widetilde{g}, v_h) = 0 \quad \text{for all } v_h \in \widetilde{X}_h. \tag{3.60}$$

Applying standard arguments we get unique solvability of (3.60), the stability estimate

$$||u_{0,h}||_{H^1(\Omega)} \le \frac{c_2^A}{c_1^A} \cdot ||\widetilde{g}||_{H^1(\Omega)} \tag{3.61}$$

and the quasi–optimal error estimate

$$||u_0 - u_{0,h}||_{H^1(\Omega)} \le \frac{c_2^A}{c_1^A} \cdot \inf_{v_h \in \widetilde{X}_h} ||u_0 - v_h||_{H^1(\Omega)} \,. \tag{3.62}$$

Now we can define an approximate Steklov–Poincaré operator $\widetilde{S}g$ by

$$\langle \widetilde{S}g, w\rangle_{L_2(\Gamma)} := a(u_{0,h} + \widetilde{g}, \mathcal{E}w) \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.63}$$

Theorem 3.5. *The approximate Steklov–Poincaré operator* $\widetilde{S}$ *defined by (3.63) is bounded,*

$$||\widetilde{S}g||_{H^{-1/2}(\Gamma)} \le c_2^{\widetilde{S}} \cdot ||g||_{H^{1/2}(\Gamma)} \quad \textit{for all } g \in H^{1/2}(\Gamma) \tag{3.64}$$

and satisfies the quasi–optimal error estimate

$$||(S - \widetilde{S})g||_{H^{-1/2}(\Gamma)} \le c \cdot \inf_{v_h \in \widetilde{X}_h} ||u_0 - v_h||_{H^1(\Omega)} \tag{3.65}$$

where $u_0 \in H_0^1(\Omega, \Gamma)$ *is the unique solution of (3.58).*

Moreover, $\widetilde{S}$ *is elliptic on* $H_0^{1/2}(\Gamma, \Gamma_D)$,

$$\langle \widetilde{S}g, g\rangle_{L_2(\Gamma)} \ge c_1^S \cdot ||g||^2_{H^{1/2}(\Gamma)} \quad \textit{for all } g \in H_0^{1/2}(\Gamma, \Gamma_D). \tag{3.66}$$

Proof. Using the norm definition in $H^{-1/2}(\Gamma)$ by duality, we get with (3.63), (3.5), (1.1) and (3.61)

$$\begin{aligned}
||\widetilde{S}g||_{H^{-1/2}(\Gamma)} &= \sup_{0 \ne w \in H^{1/2}(\Gamma)} \frac{|\langle \widetilde{S}g, w\rangle_{L_2(\Gamma)}|}{||w||_{H^{1/2}(\Gamma)}} = \sup_{0 \ne w \in H^{1/2}(\Gamma)} \frac{|a(u_{0,h} + \widetilde{g}, \widetilde{w})|}{||w||_{H^{1/2}(\Gamma)}} \\
&\le c_2^A \cdot c_{IT} \cdot ||u_{0,h} + \widetilde{g}||_{H^1(\Omega)} \le c_2^A \cdot c_{IT} \cdot (1 + \frac{c_2^A}{c_1^A}) \cdot ||\widetilde{g}||_{H^1(\Omega)}.
\end{aligned}$$

Now, (3.64) follows from the inverse trace theorem. To derive the error estimate (3.65) we apply similar ideas to get

$$\begin{aligned}
||(S - \widetilde{S})g||_{H^{-1/2}(\Gamma)} &= \sup_{0 \ne w \in H^{1/2}(\Gamma)} \frac{|\langle Sg - \widetilde{S}g, w\rangle_{L_2(\Gamma)}|}{||w||_{H^{1/2}(\Gamma)}} \\
&= \sup_{0 \ne w \in H^{1/2}(\Gamma)} \frac{|a(u_0 - u_{0,h}, \widetilde{w})|}{||w||_{H^{1/2}(\Gamma)}} \\
&\le c_2^A \cdot c_{IT} \cdot ||u_0 - u_{0,h}||_{H^1(\Omega)} \\
&\le c_2^A \cdot c_{IT} \cdot \frac{c_2^A}{c_1^A} \cdot \inf_{v_h \in \widetilde{X}_h} ||u_0 - v_h||_{H^1(\Omega)}.
\end{aligned}$$

The proof of the ellipticity estimate (3.66) follows as in the proof of Theorem 3.3, note that $u_{0,h} \in \widetilde{X}_h \subset H_0^1(\Omega, \Gamma)$. $\square$

It is important to note that in the previous theorem we only assumed that the trial space $\widetilde{X}_h \subset H_0^1(\Omega, \Gamma)$ is conform, and to ensure convergence, has to satisfy a certain approximation property.

It remains to consider a suitable approximation of the Newton potential Nf defined by (3.49). For this we consider the Galerkin equations of (3.45): find $u_h \in \widetilde{X}_h \subset H_0^1(\Omega, \Gamma)$ such that

$$a(u_h, v_h) = \int_\Omega f(x) v_h(x) dx \quad \text{for all } v_h \in \widetilde{X}_h. \tag{3.67}$$

As in (3.61) and (3.61) we have the stability estimate

$$||u_h||_{H^1(\Omega)} \leq \frac{1}{c_1^A} \cdot ||f||_{\widetilde{H}^{-1}(\Omega)} \tag{3.68}$$

and the quasi–optimal error estimate

$$||u - u_h||_{H^1(\Omega)} \leq \frac{c_2^A}{c_1^A} \cdot \inf_{v_h \in \widetilde{X}_h} ||u - v_h||_{H^1(\Omega)}\,. \tag{3.69}$$

As in (3.47) we can define an approximate conormal derivative $\widetilde{\lambda} \in H^{-1/2}(\Gamma)$ satisfying

$$\int_\Gamma \widetilde{\lambda}(x) w(x) ds_x = a(u_h, \mathcal{E}w) - \int_\Omega f(x) \mathcal{E}w(x) dx \quad \text{for all } w \in H^{1/2}(\Gamma) \tag{3.70}$$

and the approximate Newton potential is given by

$$\widetilde{N}f(x) := -\widetilde{\lambda}(x) \quad \text{for } x \in \Gamma. \tag{3.71}$$

Theorem 3.6. *The approximate Newton potential defined by (3.71) is bounded,*

$$||\widetilde{N}f||_{H^{-1/2}(\Gamma)} \leq c \cdot ||f||_{H^{-1}(\Omega)} \quad \textit{for all } f \in H^{-1}(\Omega), \tag{3.72}$$

and satisfies the quasi–optimal error estimate

$$||(N - \widetilde{N})f||_{H^{-1/2}(\Gamma)} \leq c \cdot \inf_{v_h \in \widetilde{X}_h} ||u - v_h||_{H^1(\Omega)} \tag{3.73}$$

where $u \in H_0^1(\Omega)$ is the unique solution of (3.45).

The proof of Theorem 3.6 follows as the proof of Theorem 3.5, we skip the details.

By using (3.63) and (3.71) in (3.56) we have defined an approximate Dirichlet–Neumann map (3.56) using finite element methods to approximate both the Steklov–Poincaré operator and the Newton potential. Since we are dealing with linear problems, we can combine both approximations $\widetilde{S}$ and $\widetilde{N}f$. Hence, the approximate Dirichlet–Neumann map, in particular, the approximate Neumann datum $\widetilde{\lambda} \in H^{-1/2}(\Gamma)$ satisfies

$$\int_\Gamma \widetilde{\lambda}(x)w(x)ds_x \;=\; a(u_{0,h} + \mathcal{E}g, \mathcal{E}w) - \int_\Omega f(x)\mathcal{E}w(x)dx \tag{3.74}$$

for all $w \in H^{1/2}(\Gamma)$ where $u_{0,h} \in \widetilde{X}_h \subset H^1_0(\Omega)$ solves

$$a(u_{0,h}, v_h) \;=\; \int_\Omega f(x)v_h(x)dx - a(\mathcal{E}g, v_h) \quad \text{for all } v_h \in \widetilde{X}_h. \tag{3.75}$$

Combining the error estimate (3.65) for the approximate Steklov–Poincaré operator $\widetilde{S}$ and (3.73) for the approximate Newton potential $\widetilde{N}f$ we get an error estimate for the approximate Dirichlet–Neumann map,

$$||\lambda - \widetilde{\lambda}||_{H^{-1/2}(\Gamma)} \;\leq\; c \cdot \inf_{v_h \in \widetilde{X}_h} ||u_0 - v_h||_{H^1(\Omega)}. \tag{3.76}$$

3.4 Approximation by Boundary Element Methods

In this section we describe and analyze a stable boundary element approximation of the Steklov–Poincaré operator S based on the symmetric representation (3.40). For a given $g \in H^{1/2}(\Gamma)$ the application Sg of the Steklov–Poincaré operator given by (3.40) reads for $x \in \Gamma$,

$$\begin{aligned} Sg(x) &= Dg(x) + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)g(x) \\ &= Dg(x) + (\frac{1}{2}I + K')w(x) \end{aligned}$$

where $w \in H^{-1/2}(\Gamma)$ is the unique solution of

$$\langle Vw, \tau\rangle_{L_2(\Gamma)} \;=\; \langle(\frac{1}{2}I + K)g, \tau\rangle_{L_2(\Gamma)} \quad \text{for all } \tau \in H^{-1/2}(\Gamma). \tag{3.77}$$

Note that, by using (3.39),

$$w \;=\; V^{-1}(\frac{1}{2}I + K)g \;=\; Sg\,.$$

To define an approximation $\widetilde{S}g$, let

$$Z_h \;:=\; \text{span}\{\psi_\ell\}_{\ell=1}^N \subset H^{-1/2}(\Gamma) \tag{3.78}$$

be a finite–dimensional trial space. The Galerkin formulation of (3.77) is: find $w_h \in Z_h$ such that

$$\langle V w_h, \tau_h \rangle_{L_2(\Gamma)} = \langle (\frac{1}{2} I + K) g, \tau_h \rangle_{L_2(\Gamma)} \quad \text{for all } \tau_h \in Z_h. \tag{3.79}$$

Thus,

$$\widetilde{S} g(x) := D g(x) + (\frac{1}{2} I + K') w_h(x) \tag{3.80}$$

defines an approximation $\widetilde{S} g$ of the Steklov–Poincaré operator Sg.

Theorem 3.7. *The approximate Steklov–Poincaré operator $\widetilde{S}$ defined by (3.80) is bounded,*

$$||\widetilde{S} g||_{H^{-1/2}(\Gamma)} \le c_2^{\widetilde{S}} \cdot ||g||_{H^{1/2}(\Gamma)} \quad \textit{for all } g \in H^{1/2}(\Gamma) \tag{3.81}$$

and satisfies the quasi–optimal error estimate

$$||(S - \widetilde{S}) g||_{H^{-1/2}(\Gamma)} \le c \cdot \inf_{\tau_h \in Z_h} ||Sg - \tau_h||_{H^{-1/2}(\Gamma)}. \tag{3.82}$$

Moreover, $\widetilde{S}$ is elliptic on $H_0^{1/2}(\Gamma, \Gamma_D)$,

$$\langle \widetilde{S} g, g \rangle_{L_2(\Gamma)} \ge c_1^S \cdot ||g||^2_{H^{1/2}(\Gamma)} \quad \textit{for all } g \in H_0^{1/2}(\Gamma, \Gamma_D). \tag{3.83}$$

Proof. Choosing in (3.79) $\tau_h = w_h \in Z_h$ we get

$$\begin{aligned} c_1^V \cdot ||w_h||^2_{H^{1/2}(\Gamma)} &\le \langle V w_h, w_h \rangle_{L_2(\Gamma)} \\ &= \langle (\frac{1}{2} I + K) g, w_h \rangle_{L_2(\Gamma)} \le c \cdot ||g||_{H^{1/2}(\Gamma)} ||w_h||_{H^{-1/2}(\Gamma)} \end{aligned}$$

and therefore

$$||w_h||_{H^{-1/2}(\Gamma)} \le c \cdot ||g||_{H^{1/2}(\Gamma)}.$$

Hence,

$$\begin{aligned} ||\widetilde{S} g||_{H^{-1/2}(\Gamma)} &= ||Dg + (\frac{1}{2} I + K') w_h||_{H^{-1/2}(\Gamma)} \\ &\le c \cdot \left\{ ||g||_{H^{1/2}(\Gamma)} + ||w_h||_{H^{-1/2}(\Gamma)} \right\} \le \widetilde{c} \cdot ||g||_{H^{1/2}(\Gamma)} . \end{aligned}$$

Applying standard arguments, in particular Cea's lemma, we get for the Galerkin solution $w_h \in Z_h$ of (3.79) the quasi–optimal error estimate

$$||w - w_h||_{H^{-1/2}(\Gamma)} \le c \cdot \inf_{\tau_h \in Z_h} ||w - \tau_h||_{H^{-1/2}(\Gamma)}.$$

Then,

$$||(S-\widetilde{S})g||_{H^{-1/2}(\Gamma)} = ||(\frac{1}{2}I+K')(w-w_h)||_{H^{-1/2}(\Gamma)}$$
$$\leq c\cdot ||w-w_h||_{H^{-1/2}(\Gamma)} \leq \widetilde{c}\cdot \inf_{\tau_h\in Z_h} ||w-\tau_h||_{H^{-1/2}(\Gamma)}.$$

Using $w = Sg$, (3.82) follows. Now let $g \in H_0^{1/2}(\Gamma, \Gamma_D)$. Using the definition (3.80) of $\widetilde{S}g$, the Galerkin formulation (3.79) and the ellipticity (3.35) of the single layer potential we get

$$\begin{aligned}\langle\widetilde{S}g, g\rangle_{L_2(\Gamma)} &= \langle\langle Dg, g\rangle_{L_2(\Gamma)} + \langle(\frac{1}{2}I+K')w_h, g\rangle_{L_2(\Gamma)} \\ &= \langle\langle Dg, g\rangle_{L_2(\Gamma)} + \langle(w_h, (\frac{1}{2}I+K)g\rangle_{L_2(\Gamma)} \\ &= \langle Dg, g\rangle_{L_2(\Gamma)} + \langle Vw_h, w_h\rangle_{L_2(\Gamma)} \geq \langle Dg, g\rangle_{L_2(\Gamma)}\end{aligned}$$

and (3.66) follows from (3.36). □

Note that the result of the previous theorem corresponds to the statement of Theorem 3.5 in the case of a finite element approximation. In both cases no further conditions on the definition of the trial spaces $\widetilde{X}_h$ and Z_h have to be required, only some approximation properties have to be assumed.

Instead of the symmetric approximation (3.80) of the Steklov–Poincaré operator we may use any other stable approximation, which is based on an equivalent boundary integral representation of the Steklov–Poincaré operator [63]. In particular, one can use a hybrid discretization [64] of

$$S = V^{-1}(\frac{1}{2}I+K)VV^{-1} = V^{-1}FV^{-1}$$

leading to a symmetric stiffness matrix or one can use a mixed discretization [65] of

$$S = V^{-1}(\frac{1}{2}I+K),$$

which leads to a nonsymmetric stiffness matrix even for a self–adjoint operator S. Note that in both cases appropriate discrete inf–sup conditions as described in Chapter 2 are needed to ensure stability.

Let us finally consider a boundary element approximation of the Newton potential (3.55). This is equivalent to find $\lambda \in H^{-1/2}(\Gamma)$ such that

$$\langle V\lambda, \mu\rangle_{L_2(\Gamma)} = -\langle N_0 f, \mu\rangle_{L_2(\Gamma)} \quad \text{for all } \mu \in H^{-1/2}(\Gamma). \tag{3.84}$$

Hence, to define an approximate Newton potential $\widetilde{N}f$ we may consider the Galerkin variational problem: find $\lambda_h \in Z_h$ such that

$$\langle V\lambda_h, \mu_h\rangle_{L_2(\Gamma)} = -\langle N_0 f, \mu_h\rangle_{L_2(\Gamma)} \quad \text{for all } \mu_h \in Z_h. \tag{3.85}$$

Now we can define

$$\widetilde{N}f(x) := \lambda_h(x) \in Z_h \subset H^{-1/2}(\Gamma). \tag{3.86}$$

Applying standard arguments we get the stability estimate

$$||\widetilde{N}f||_{H^{-1/2}(\Gamma)} \leq c \cdot ||f||_{H^{-1}(\Omega)} \tag{3.87}$$

as well as the quasi–optimal error estimate

$$||(N - \widetilde{N})f||_{H^{-1/2}(\Gamma)} \leq c \cdot \inf_{\mu_h \in Z_h} ||\lambda - \mu_h||_{H^{-1/2}(\Gamma)} \tag{3.88}$$

where $\lambda \in H^{-1/2}(\Gamma)$ is the unique solution of (3.84).

When solving the Galerkin problem (3.85) to compute the approximate Newton potential $\widetilde{N}f = \lambda_h$, we need to evaluate the right hand side for $\mu_h = \psi_\ell$ and $\ell = 1, \ldots, N$,

$$f_\ell := -\int_\Gamma \psi_\ell(x) \int_\Omega U^*(x,y) f(y) dy ds_x .$$

Hence we need to have some triangulation of Ω as well. To avoid this drawback, one can approximate the volume integral as follows: Let us consider the partial differential equation (with constant coefficients)

$$Lu(x) = f(x) \quad \text{for } x \in \Omega \tag{3.89}$$

whose solution is given by the representation formula for $x \in \Omega$,

$$u(x) = \int_\Gamma U^*(x,y)\gamma_1 u(y) ds_y - \int_\Gamma \gamma_1(y) U^*(x,y)\gamma_0 u(y) ds_y + \int_\Omega U^*(x,y) f(y) dy.$$

Applying the trace operator γ_0 gives

$$N_0 f(x) = (\frac{1}{2}I + K)\gamma_0 u - V\gamma_1 u(x) \quad \text{for } x \in \Gamma. \tag{3.90}$$

Note that (3.90) holds for any pair $[\gamma_0 u, \gamma_1 u]$ of Cauchy–data where u is a solution of the partial differential equation (3.89). Hence, to compute (3.90) it is sufficient to have at least one particular solution u_p of the partial differential equation (3.89) to be inserted in (3.90). Instead of (3.89) we now consider the extended boundary value problem

$$L\widetilde{u}(x) = \widetilde{f}(x) \quad \text{for } x \in \Omega_0, \quad \widetilde{u}(x) = 0 \quad \text{for } x \in \partial\Omega_0, \tag{3.91}$$

where $\Omega_0 \supset \Omega$ is some fictitious domain. Here, $\widetilde{f} \in L_2(\Omega_0)$ is some extension of the given data $f \in L_2(\Omega)$, for example by zero. The unique solution $\widetilde{u} \in H_0^1(\Omega_0)$ of (3.91) is a particular solution $\widetilde{u}_{|\Omega} \in H^1(\Omega)$ of the partial differential equation (3.89). Using a finite element method to solve (3.91) numerically we can define a suitable approximation of the Newton potential

(3.90), in particular when inserting the finite element solution $\widetilde{u}_h$. Hence we need to compute the traces of $\widetilde{u}_h$ on $\Gamma = \partial\Omega$ efficiently. Using a hierarchical triangulation of the fictitious domain Ω_0 we can solve the related finite element system using multilevel preconditioners as well as we can find pointwise values of the approximate solution efficiently. To approximate $\nabla\widetilde{u}_h$ one can use an L_2 projection onto a continuous finite element space. For a complete description of this algorithm we refer to [48, 62].

4

Mixed Discretization Schemes

In this chapter we will consider hybrid formulations for the mixed boundary value problem (3.1) which are based on the Dirichlet–Neumann map (3.16). The mixed boundary value problem (3.1) is obviously equivalent to the coupled problem to find $(u,\lambda) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$:

$$\begin{aligned} \lambda(x) &= Su(x) - Nf(x) && \text{for } x \in \Gamma, \\ u(x) &= g_D(x) && \text{for } x \in \Gamma_D, \\ \lambda(x) &= g_N(x) && \text{for } x \in \Gamma_N. \end{aligned} \tag{4.1}$$

The standard variational formulation of (4.1) is: find $u \in H^{1/2}(\Gamma)$, $u(x) = g_D(x)$ for $x \in \Gamma_D$ such that

$$\int_\Gamma Su(x)v(x)ds_x = \int_\Omega Nf(x)v(x)ds_x + \int_{\Gamma_N} g_N(x)v(x)ds_x \tag{4.2}$$

for all $v \in H_0^{1/2}(\Gamma,\Gamma_D)$. Since the Steklov–Poincaré operator S is elliptic on $H_0^{1/2}(\Gamma,\Gamma_D)$, there exists a unique solution $u \in H^{1/2}(\Gamma)$ of (4.2). For practical computations, however, we have to replace both the Steklov–Poincaré operator S and the Newton potential Nf in (4.1) by some approximations $\widetilde{S}$ and $\widetilde{N}f$ as introduced in the previous chapter. We assume in general, that these approximations are bounded and satisfy some approximation properties. Moreover, the approximate Steklov–Poincaré operators $\widetilde{S}$ are assumed to be elliptic on suitable subspaces. Hence, instead of (4.1) we have to consider a coupled problem to find $(\widetilde{u},\widetilde{\lambda}) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$:

$$\begin{aligned} \widetilde{\lambda}(x) &= \widetilde{S}\widetilde{u}(x) - \widetilde{N}f(x) && \text{for } x \in \Gamma, \\ \widetilde{u}(x) &= g_D(x) && \text{for } x \in \Gamma_D, \\ \widetilde{\lambda}(x) &= g_N(x) && \text{for } x \in \Gamma_N. \end{aligned} \tag{4.3}$$

O. Steinbach: LNM 1809, pp. 71–83, 2003.

To derive hybrid discretization schemes for (4.3) one may consider either a strong or a weak formulation of the Dirichlet boundary conditions in (4.3). Note that the latter will lead to a formulation with Lagrange multipliers.

4.1 Variational Methods with Approximate Steklov–Poincaré Operators

We start with considering the standard variational formulation for (4.3) by including the Dirichlet boundary conditions in strong form and the Neumann boundary conditions in a weak one. So we have to find $\widetilde{u} \in H^{1/2}(\Gamma)$ with $\widetilde{u}(x) = g_D(x)$ for $x \in \Gamma_D$ such that

$$\int_\Gamma \widetilde{S}\widetilde{u}(x)v(x)ds_x = \int_\Gamma \widetilde{N}f(x)v(x)ds_x + \int_{\Gamma_N} g_N(x)v(x)ds_x \tag{4.4}$$

for all $v \in H^{1/2}(\Gamma)$ with $v(x) = 0$ for $x \in \Gamma_D$.

Theorem 4.1. *Let $\widetilde{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ be bounded and elliptic on $H_0^{1/2}(\Gamma, \Gamma_D)$. Then there exists a unique solution $\widetilde{u} \in H^{1/2}(\Gamma)$ of (4.4) satisfying the error estimate*

$$||u - \widetilde{u}||_{H^{1/2}(\Gamma)} \le \frac{1}{c_1^S} \cdot \Big[||(S - \widetilde{S})u||_{H^{-1/2}(\Gamma)} + ||(N - \widetilde{N})f||_{H^{-1/2}(\Gamma)} \Big] \tag{4.5}$$

where $u \in H^{1/2}(\Gamma)$ is the unique solution of (4.2).

Proof. Since the approximate Steklov–Poincaré operator $\widetilde{S}$ is assumed to be bounded and elliptic, the unique solvability of (4.4) follows by applying the Lax–Milgram theorem. So it remains to prove (4.5). Subtracting (4.4) from (4.2) we get the variational equality

$$\langle Su - \widetilde{S}\widetilde{u}, v\rangle_{L_2(\Gamma)} = \langle Nf - \widetilde{N}f, v\rangle_{L_2(\Gamma)} \quad \text{for all } v \in H_0^{1/2}(\Gamma, \Gamma_D)\,.$$

Note that $u - \widetilde{u} \in H_0^{1/2}(\Gamma, \Gamma_D)$. Hence we have

$$\begin{aligned} c_1^S \cdot ||u - \widetilde{u}||^2_{H^{1/2}(\Gamma)} &\le \langle \widetilde{S}(u - \widetilde{u}), u - \widetilde{u}\rangle_{L_2(\Gamma)} \\ &= \langle (\widetilde{S} - S)u, u - \widetilde{u}\rangle_{L_2(\Gamma)} + \langle (N - \widetilde{N})f, u - \widetilde{u}\rangle_{L_2(\Gamma)} \\ &\le \Big[||(S - \widetilde{S})u||_{H^{-1/2}(\Gamma)} + ||(N - \widetilde{N})f||_{H^{-1/2}(\Gamma)} \Big] ||u - \widetilde{u}||_{H^{1/2}(\Gamma)} \end{aligned}$$

and the assertion follows. □

Therefore, in the case of a finite element approximation we get as resulting error estimate by combining (3.65) and (3.73),

$$||u - \widetilde{u}||_{H^{1/2}(\Gamma)} \le c \cdot \inf_{v_h \in \widetilde{X}_h} ||u - v_h||_{H^1(\Omega)}. \tag{4.6}$$

When using a boundary element approximation for the Steklov–Poincaré operator and the Newton potential, with (3.82) and (3.88) we get

$$||u - \widetilde{u}||_{H^{1/2}(\Gamma)} \le c \cdot \inf_{\mu_h \in Z_h} ||\lambda - \mu_h||_{H^{-1/2}(\Gamma)}. \tag{4.7}$$

Now we consider a Galerkin approach to solve the perturbed variational problem (4.4). Let $\widetilde{g}_D \in H^{1/2}(\Gamma)$ be an arbitrary but fixed extension of the given Dirichlet data with $\widetilde{g}_D(x) = g_D(x)$ for $x \in \Gamma_D$. Then we have to find $\widetilde{u}_0 \in H_0^{1/2}(\Gamma, \Gamma_D)$ such that

$$\langle \widetilde{S}\widetilde{u}_0, v\rangle_{L_2(\Gamma)} = \langle \widetilde{N}f + g_N - \widetilde{S}\widetilde{g}_D, v\rangle_{L_2(\Gamma)} \quad \text{for all } v \in H_0^{1/2}(\Gamma, \Gamma_D). \tag{4.8}$$

Let

$$X_h := \text{span}\{\varphi_k\}_{k=1}^M \subset H_0^{1/2}(\Gamma, \Gamma_D)$$

be a finite–dimensional trial space, then the Galerkin variational problem of (4.8) is: find $\widetilde{u}_{0,h} \in X_h$ such that

$$\langle \widetilde{S}\widetilde{u}_{0,h}, w_h\rangle_{L_2(\Gamma)} = \langle \widetilde{N}f + g_N - \widetilde{S}\widetilde{g}_D, w_h\rangle_{L_2(\Gamma)} \quad \text{for all } w_h \in X_h. \tag{4.9}$$

Then, the Galerkin approximation of $\widetilde{u}$ is defined by $\widetilde{u}_h := \widetilde{u}_{0,h} + \widetilde{g}_D$.

Theorem 4.2. *There exists a unique solution $\widetilde{u}_{0,h} \in X_h$ of (4.9) satisfying the quasi–optimal error estimate*

$$||u-\widetilde{u}_h||_{H^{1/2}(\Gamma)} \le c_1 \cdot \inf_{v_h \in X_h} ||(u-\widetilde{g}_D)-v_h||_{H^{1/2}(\Gamma)} + c_2 \cdot ||u-\widetilde{u}||_{H^{1/2}(\Gamma)}. \tag{4.10}$$

where $u \in H^{1/2}(\Gamma)$ is the unique solution of (4.2).

Proof. Since the approximate Steklov–Poincaré operator $\widetilde{S}$ is elliptic on $H_0^{1/2}(\Gamma, \Gamma_D)$, the unique solvability of (4.9) follows by applying standard arguments. Moreover, there holds the error estimate

$$||\widetilde{u}_0 - \widetilde{u}_{0,h}||_{H^{1/2}(\Gamma)} \le c \cdot \inf_{w_h \in X_h} ||\widetilde{u}_0 - w_h||_{H^{1/2}(\Gamma)} .$$

Note that

$$\begin{aligned} ||\widetilde{u}_0 - w_h||_{H^{1/2}(\Gamma)} &= ||\widetilde{u}_0 + \widetilde{g}_D - u + (u - \widetilde{g}_D) - w_h||_{H^{1/2}(\Gamma)} \\ &\le ||u - \widetilde{u}||_{H^{1/2}(\Gamma)} + ||(u - \widetilde{g}_D) - w_h||_{H^{1/2}(\Gamma)} \end{aligned}$$

and therefore

$$||\widetilde{u}_0 - \widetilde{u}_{0,h}||_{H^{1/2}(\Gamma)} \le c \cdot \left\{ ||u - \widetilde{u}||_{H^{1/2}(\Gamma)} + \inf_{w_h \in X_h} ||(u - \widetilde{g}_D) - w_h||_{H^{1/2}(\Gamma)} \right\}.$$

Now,

$$
\begin{aligned}
||u - \widetilde{u}_h||_{H^{1/2}(\Gamma)} &\le ||u - \widetilde{u}||_{H^{1/2}(\Gamma)} + ||\widetilde{u} - \widetilde{u}_h||_{H^{1/2}(\Gamma)} \\
&= ||u - \widetilde{u}||_{H^{1/2}(\Gamma)} + ||\widetilde{u}_0 - \widetilde{u}_{0,h}||_{H^{1/2}(\Gamma)} \\
&\le (1+c) \cdot ||u - \widetilde{u}||_{H^{1/2}(\Gamma)} + c \cdot \inf_{v_h \in X_h} ||(u - \widetilde{g}_D) - v_h||_{H^{1/2}(\Gamma)}. \square
\end{aligned}
$$

The Galerkin equations (4.9) are equivalent to a system of linear equations,

$$\widetilde{S}_h \underline{\widetilde{u}}_0 = \underline{f}, \tag{4.11}$$

where the stiffness matrix is given by

$$\widetilde{S}_h[\ell, k] \;=\; \langle \widetilde{S}\varphi_k, \varphi_\ell \rangle_{L_2(\Gamma)} \tag{4.12}$$

for $k, \ell = 1, \ldots, M$. Note that the Galerkin matrix of the Steklov–Poincaré operator S is given by

$$S_h[\ell, k] \;=\; \langle S\varphi_k, \varphi_\ell \rangle_{L_2(\Gamma)} \quad \text{for } k, \ell = 1, \ldots, M$$

where the Steklov–Poincaré operator S is given either by (3.24) or by (3.40). First we will describe the matrix representation of $\widetilde{S}_h$ when using the approximate Steklov–Poincaré operator $\widetilde{S}$ as defined by the finite element approximation (3.63):

Using the approximate Dirichlet–Neumann map (3.74) to replace in (4.9) the approximations $\widetilde{S}$ and $\widetilde{N}f$, the Galerkin problem (4.9) reads: find $\widetilde{u}_{0,h} \in X_h$ such that

$$a(\mathcal{E}\widetilde{u}_{0,h} + \mathcal{E}\widetilde{g}_D + u_{0,h}, \mathcal{E}w_h) \;=\; \int\limits_{\Omega} f(x)\mathcal{E}w_h(x)ds_x + \int\limits_{\Gamma_N} g_N(x)w_h(x)ds_x \tag{4.13}$$

for all $w_h \in X_h$ where $u_{0,h} \in \widetilde{X}_h$ solves

$$a(u_{0,h} + \mathcal{E}\widetilde{u}_{0,h} + \mathcal{E}\widetilde{g}_D, v_h) \;=\; \int\limits_{\Omega} f(x)v_h(x)dx \tag{4.14}$$

for all $v_h \in \widetilde{X}_h$. For $k, \ell = 1, \ldots, M$ and $i, j = 1, \ldots, \widetilde{M}$ we define

$$
\begin{aligned}
A_{h,\Omega,\Omega}[j, i] &= a(\phi_i, \phi_j), \\
A_{h,\Gamma,\Gamma}[\ell, k] &= a(\mathcal{E}\varphi_k, \mathcal{E}\varphi_\ell), \\
A_{h,\Gamma,\Omega}[j, k] &= a(\mathcal{E}\varphi_k, \phi_j)
\end{aligned}
$$

as well as

$$f_{\Omega,j} := \int_\Omega f(x)\phi_j(x)dx - a(\mathcal{E}\widetilde{g}_D, \phi_j),$$

$$f_{\Gamma,\ell} := \int_\Omega f(x)\mathcal{E}\varphi_\ell(x)dx + \int_{\Gamma_N} g_N(x)\varphi_\ell(x)ds_x - a(\mathcal{E}\widetilde{g}_D, \mathcal{E}\varphi_\ell).$$

Then, the Galerkin equations (4.13) and (4.14) are equivalent to the linear system

$$\begin{pmatrix} A_{h,\Omega,\Omega} & A_{h,\Gamma,\Omega} \\ A_{h,\Gamma,\Omega}^\top & A_{h,\Gamma,\Gamma} \end{pmatrix} \begin{pmatrix} \underline{u}_0 \\ \underline{\widetilde{u}}_0 \end{pmatrix} = \begin{pmatrix} \underline{f}_\Omega \\ \underline{f}_\Gamma \end{pmatrix}. \tag{4.15}$$

Since the matrix $A_{h,\Omega,\Omega}$ is invertible we can eliminate $\underline{u}_0$ to get the Schur complement system

$$\left[A_{h,\Gamma,\Gamma} - A_{h,\Gamma,\Omega}^\top A_{h,\Omega,\Omega}^{-1} A_{h,\Gamma,\Omega}\right] \underline{\widetilde{u}}_0 = \underline{f}_\Gamma - A_{h,\Gamma,\Omega}^\top A_{h,\Omega,\Omega}^{-1} \underline{f}_\Omega \,. \tag{4.16}$$

Note that (4.16) corresponds to (4.11), in particular,

$$\widetilde{S}_h^{FEM} := A_{h,\Gamma,\Gamma} - A_{h,\Gamma,\Omega}^\top A_{h,\Omega,\Omega}^{-1} A_{h,\Gamma,\Omega} \tag{4.17}$$

is the discrete Steklov–Poincaré operator based on finite elements.

Lemma 4.3. *Let $X_h \subset H_0^{1/2}(\Gamma,\Gamma_D)$. Then,*

$$(S_h\underline{u},\underline{u}) \le (\widetilde{S}_h^{\mathrm{FEM}}\underline{u},\underline{u}) \le \frac{c_2^A c_{IT}^2}{c_1^S} \cdot (S_h\underline{u},\underline{u}) \quad \textit{for all } \underline{u} \in \mathbb{R}^M \leftrightarrow u_h \in X_h.$$

Proof. For $\underline{u} \in \mathbb{R}^M \leftrightarrow u_h \in X_h \subset H_0^{1/2}(\Gamma,\Gamma_D)$ we have

$$(\widetilde{S}_h^{\mathrm{FEM}}\underline{u},\underline{u}) = a(\mathcal{E}u_h + u_{0,h}, \mathcal{E}u_h)$$

where $u_{0,h} \in \widetilde{X}_h \subset H_0^1(\Omega)$ solves

$$a(u_{0,h} + \mathcal{E}u_h, v_h) = 0 \quad \text{for all } v_h \in \widetilde{X}_h.$$

In a similar way we get

$$(S_h\underline{u},\underline{u}) = a(\mathcal{E}u_h + u_0, \mathcal{E}u_h)$$

where $u_0 \in H_0^1(\Omega)$ solves

$$a(u_0 + \mathcal{E}u_h, v) = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

Hence,

$$a(u_{0,h}, v_h) = a(u_0, v_h) \quad \text{for all } v_h \in \widetilde{X}_h$$

from which

$$a(u_{0,h}, u_{0,h}) \leq a(u_0, u_0)$$

follows. Then,

$$\begin{aligned}(\widetilde{S}_h^{\rm FEM}\underline{u}, \underline{u}) &= a(\mathcal{E}u_h + u_{0,h}, \mathcal{E}u_h) = a(\mathcal{E}u_h, \mathcal{E}u_h) - a(u_{0,h}, u_{0,h}) \\ &\geq a(\mathcal{E}u_h, \mathcal{E}u_h) - a(u_0, u_0) = a(\mathcal{E}u_h + u_0, \mathcal{E}u_h) = (S_h\underline{u}, \underline{u}).\end{aligned}$$

The upper estimate follows by using the positive definitness of $A_{h,\Omega,\Omega}$, the boundedness of the bilinear form $a(\cdot,\cdot)$ and of the extension operator $\mathcal{E}$ and the ellipticity of S:

$$\begin{aligned}(\widetilde{S}_h^{\rm FEM}\underline{u}, \underline{u}) &= (A_{h,\Gamma,\Gamma}\underline{u}, \underline{u}) - (A_{h,\Omega,\Omega}^{-1}A_{h,\Gamma,\Omega}\underline{u}, A_{h,\Gamma,\Omega}\underline{u}) \leq (A_{h,\Gamma,\Gamma}\underline{u}, \underline{u}) \\ &= a(\mathcal{E}u_h, \mathcal{E}u_h) \leq c_2^A \cdot ||\mathcal{E}u_h||^2_{H^1(\Omega)} \leq c_2^A c_{IT}^2 \cdot ||u||^2_{H^{1/2}(\Gamma)} \\ &= c_2^A c_{IT}^2 / c_1^S \cdot \langle Su_h, u_h\rangle_{L_2(\Gamma)} = c_2^A c_{IT}^2/c_1^S \cdot (S_h\underline{u}, \underline{u}). \qquad \square\end{aligned}$$

Note that the spectral equivalence inequalities for $\widetilde{S}_h^{FEM}$ and S_h hold for an arbitrary choice of the finite element space $\widetilde{X}_h \subset H_0^1(\Omega)$ but depend on the extension operator $\mathcal{E} : H^{1/2}(\Gamma) \to H^1(\Omega)$.

It remains to describe the (discrete) extension operator $\mathcal{E}$ needed in the computations above. Let $\widetilde{X}_h^* \subset H^1(\Omega)$ be a finite element space satisfying the following matching condition: For each $w_h \in X_h$ there exists a $\widetilde{w}_h^* \in \widetilde{X}_h^*$ such that

$$\widetilde{w}_h^*(x) = w_h(x) \quad \text{for } x \in \Gamma \tag{4.18}$$

and which satisfies the variational problem

$$a(\widetilde{w}_h^*, v_h^*) = 0 \quad \text{for all } v_h^* \in \widetilde{X}_h^* \cap H_0^1(\Omega).$$

In fact, $\mathcal{E}w_h := \widetilde{w}_h^* \in \widetilde{X}_h^*$ is the discrete harmonic extension of the boundary data $w_h \in X_h$ satisfying

$$||\mathcal{E}w_h||_{H^1(\Omega)} \leq c_{IT} \cdot ||w_h||_{H^{1/2}(\Gamma)}.$$

Discrete harmonic extension operators are also used in the construction of preconditioners in domain decomposition methods, see for example [42]. Note that $\widetilde{X}_h^*$ is an extension of the trial space X_h from the boundary Γ to the domain Ω, see Figure 4.1. Especially, when the underlying mesh of the trial space $\widetilde{X}_h$ matches the mesh of the trial space X_h on the boundary Γ, one can use $\widetilde{X}_h^* = \widetilde{X}_h$. This corresponds to an interpolation of a coarse grid function on the boundary by a fine grid function in the domain Ω, see Figure 4.2.

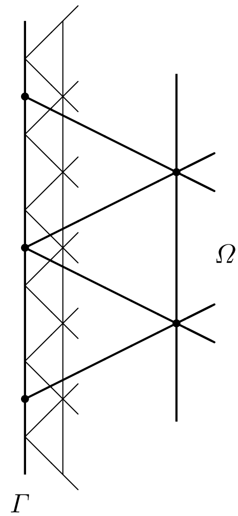


Figure 4.1:
Extension of a coarse
grid function

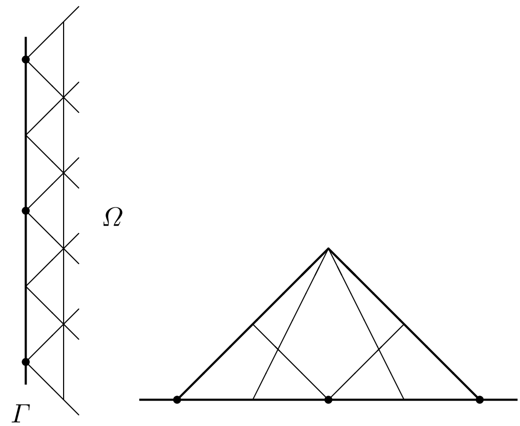


Figure 4.2:
Interpolation of a coarse grid function
on the fine grid

To describe the stiffness matrix $\widetilde{S}_h$ in the case of the approximate Steklov–Poincaré operator (3.80) when using a boundary element method we have

$$\widetilde{S}_h[\ell,k] \;=\; \langle \widetilde{S}\varphi_k,\varphi_\ell\rangle_{L_2(\Gamma)} \;=\; \langle D\varphi_k,\varphi_\ell\rangle_{L_2(\Gamma)}+\langle(\frac{1}{2}I+K')w_h^k,\varphi_\ell\rangle_{L_2(\Gamma)} \quad (4.19)$$

for $k,\ell=1,\ldots,M$ where $w_h^k\in Z_h$ is the unique solution of (3.79) with $g=\varphi_k$. Now we define for $k,\ell=1,\ldots,M$ and $i,j=1,\ldots,N$,

$$D_h[\ell,k]=\langle D\varphi_k,\varphi_\ell\rangle_{L_2(\Gamma)}, \qquad K_h[j,k]=\langle K\varphi_k,\psi_\ell\rangle_{L_2(\Gamma)},$$
$$V_h[j,i]=\langle V\psi_i,\psi_j\rangle_{L_2(\Gamma)}, \qquad M_h[j,k]=\langle \varphi_k,\psi_j\rangle_{L_2(\Gamma)}.$$

Then the Galerkin matrix $\widetilde{S}_h$ of the approximate Steklov–Poincaré operator $\widetilde{S}$ defined by (3.80) is given by

$$\widetilde{S}_h^{\mathrm{BEM}} \;=\; D_h+(\frac{1}{2}M_h^\top+K_h^\top)V_h^{-1}(\frac{1}{2}M_h+K_h)\,. \quad (4.20)$$

Note that the stiffness matrix $\widetilde{S}_h$ is symmetric and, due to (3.66), positive definite whenever $X_h\subset H_0^{1/2}(\Gamma,\Gamma_D)$ is satisfied. In particular we have the following spectral equivalence inequalities, see [27, 45], which hold for an arbitrary choice of $Z_h\subset H^{-1/2}(\Gamma)$:

Lemma 4.4. *Let $X_h\subset H^{1/2}(\Gamma,\Gamma_D)$. Then,*

$$\frac{c_1^D}{c_2^S}\cdot(S_h\underline{u},\underline{u}) \;\le\; (\widetilde{S}_h^{\mathrm{BEM}}\underline{u},\underline{u}) \;\le\; (S_h\underline{u},\underline{u}) \quad \textit{for all } \underline{u}\in\mathbb{R}^M\leftrightarrow u_h\in X_h.$$

4.2 Lagrange Multiplier Methods

In this section we consider a strong formulation of the Neumann boundary condition in (4.3) while the Dirichlet boundary conditions are formulated in a weak sense. Then we have to find $\widetilde{u} \in H^{1/2}(\Gamma)$ and $\widetilde{\lambda} \in \widetilde{H}^{-1/2}(\Gamma_D)$ such that

$$\begin{aligned} \int_\Gamma \widetilde{S}\widetilde{u}(x)v(x)ds_x - \int_{\Gamma_D} \widetilde{\lambda}(x)v(x)ds_x &= \int_{\Gamma_N} g_N(x)v(x)ds_x + \int_\Gamma \widetilde{N}f(x)v(x)ds_x \\ \int_{\Gamma_D} \widetilde{u}(x)\mu(x)ds_x &= \int_{\Gamma_D} g_D(x)\mu(x)ds_x \end{aligned} \tag{4.21}$$

for all $v \in H^{1/2}(\Gamma)$ and $\mu \in \widetilde{H}^{-1/2}(\Gamma_D)$. Defining the function spaces

$$X := H^{1/2}(\Gamma), \quad \Pi := \widetilde{H}^{-1/2}(\Gamma_D)$$

and the bounded bilinear forms

$$\begin{aligned} a(u,v) &:= \int_\Gamma \widetilde{S}u(x)v(x)ds_x : X \times X \to \mathbb{R} \\ b(u,\mu) &:= \int_{\Gamma_D} u(x)\mu(x)ds_x \quad : X \times \Pi \to \mathbb{R} \end{aligned}$$

the saddle point problem (4.21) corresponds to the abstract saddle point problem (1.13). Note that the linear forms are given by

$$\begin{aligned} \langle f, v\rangle &:= \int_{\Gamma_N} g_N(x)v(x)ds_x + \int_\Gamma \widetilde{N}f(x)v(x)ds_x : X \to \mathbb{R} \\ \langle g, \mu\rangle &:= \int_{\Gamma_D} g_D(x)\mu(x)ds_x \qquad\qquad : \Pi \to \mathbb{R}. \end{aligned}$$

It is easy to check that

$$V := \ker B = H_0^{1/2}(\Gamma, \Gamma_D)\,. \tag{4.22}$$

The bilinear form $a(\cdot,\cdot)$ is therefore elliptic on $V = \ker B$,

$$a(v,v) = \int_\Gamma \widetilde{S}v(x)v(x)ds_x \geq c_1^S \cdot ||v||^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in V = \ker B\,. \tag{4.23}$$

Hence, to apply Theorem 1.2 it remains to check the inf–sup condition (1.18) for the bilinear form $b(\cdot,\cdot)$ as defined above.

Lemma 4.5. *The bilinear form*

$$b(v,\mu) \;=\; \int\limits_{\Gamma_D} v(x)\mu(x)ds_x \;:\; H^{1/2}(\Gamma)\times \widetilde{H}^{-1/2}(\Gamma_D)$$

satisfies the inf–sup condition (1.18) with $\gamma_S = 1$.

Proof. For an arbitrary but fixed $\mu \in \widetilde{H}^{-1/2}(\Gamma_D)$ we define $u^* \in H^{1/2}(\Gamma)$ satisfying

$$\langle u^*, v\rangle_{H^{1/2}(\Gamma)} \;=\; \langle \mu, v_{|\Gamma_D}\rangle_{L_2(\Gamma_D)} \quad \text{for all } v\in H^{1/2}(\Gamma).$$

Choosing $v = u^* \in H^{1/2}(\Gamma)$ we get

$$||u^*||^2_{H^{1/2}(\Gamma)} \;=\; \langle \mu, u^*_{|\Gamma_D}\rangle_{L_2(\Gamma_D)} \;\le\; ||\mu||_{\widetilde{H}^{-1/2}(\Gamma_D)}||u^*_{|\Gamma_D}||_{H^{1/2}(\Gamma_D)}.$$

Since

$$||u^*_{|\Gamma_D}||_{H^{1/2}(\Gamma_D)} \;=\; \inf_{U\in H^{1/2}(\Gamma), U_{|\Gamma_D}=u^*} ||U||_{H^{1/2}(\Gamma)} \;\le\; ||u^*||_{H^{1/2}(\Gamma)},$$

we conclude

$$||u^*||_{H^{1/2}(\Gamma)} \;\le\; ||\mu||_{\widetilde{H}^{-1/2}(\Gamma_D)}$$

On the other hand we have by definition,

$$\begin{aligned}
||\mu||_{\widetilde{H}^{-1/2}(\Gamma_D)} &= \sup_{0\neq v\in H^{1/2}(\Gamma_D)} \frac{\langle \mu, v\rangle_{L_2(\Gamma_D)}}{||v||_{H^{1/2}(\Gamma_D)}} \\
&= \sup_{0\neq v\in H^{1/2}(\Gamma_D)} \frac{\langle \mu, v\rangle_{L_2(\Gamma_D)}}{\inf\limits_{V\in H^{1/2}(\Gamma), V_{|\Gamma_D}=v} ||V||_{H^{1/2}(\Gamma)}} \\
&= \sup_{0\neq v\in H^{1/2}(\Gamma_D)} \sup_{V\in H^{1/2}(\Gamma), V_{|\Gamma_D}=v} \frac{\langle \mu, V_{|\Gamma_D}\rangle_{L_2(\Gamma_D)}}{||V||_{H^{1/2}(\Gamma)}} \\
&= \sup_{0\neq v\in H^{1/2}(\Gamma_D)} \sup_{V\in H^{1/2}(\Gamma), V_{|\Gamma_D}=v} \frac{\langle u^*, V\rangle_{H^{1/2}(\Gamma)}}{||V||_{H^{1/2}(\Gamma)}} \;\le\; ||u^*||_{H^{1/2}(\Gamma)}.
\end{aligned}$$

Therefore we have

$$||u^*||_{H^{1/2}(\Gamma)} \;=\; ||\mu||_{\widetilde{H}^{-1/2}(\Gamma_D)}.$$

Now,

$$b(u^*,\mu) \;=\; \langle u^*_{|\Gamma_D}, \mu\rangle_{L_2(\Gamma_D)} \;=\; ||u^*||^2_{H^{1/2}(\Gamma)} \;=\; ||u^*||_{H^{1/2}(\Gamma)}||\mu||_{\widetilde{H}^{-1/2}(\Gamma_D)}$$

implying the inf–sup condition (1.18). □

Applying Theorem 1.2 there exists a unique solution $(\widetilde{u},\widetilde{\lambda}) \in H^{1/2}(\Gamma)\times \widetilde{H}^{-1/2}(\Gamma_D)$ of (4.21).

Theorem 4.6. *Let* $(u,\lambda) \in H^{1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma_D)$ *be the exact Cauchy data of (3.1). Then there holds the error estimate for the solution* $(\widetilde{u},\widetilde{\lambda})$ *of (4.21):*

$$||u-\widetilde{u}||_{H^{1/2}(\Gamma)} + ||\lambda-\widetilde{\lambda}||_{\widetilde{H}^{-1/2}(\Gamma_D)} \\ \le c \cdot \Big\{ ||(S-\widetilde{S})u||_{H^{1/2}(\Gamma)} + ||(N-\widetilde{N})f||_{H^{-1/2}(\Gamma)} \Big\}. \tag{4.24}$$

Proof. First we note that $(u,\lambda) \in H^{1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma_D)$ is the unique solution of the saddle point problem

$$\begin{aligned} \langle Su, v\rangle_{L_2(\Gamma)} - \langle v,\lambda\rangle_{L_2(\Gamma_D)} &= \langle v, g_N\rangle_{L_2(\Gamma_N)} + \langle Nf, v\rangle_{L_2(\Gamma)} \\ \langle u,\mu\rangle_{L_2(\Gamma_D)} &= \langle g_D,\mu\rangle_{L_2(\Gamma_D)} \end{aligned}$$

for all $(v,\mu) \in H^{1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma_D)$. Subtracting (4.21) from these equations we get the orthogonality relations

$$\begin{aligned} \langle Su - \widetilde{S}\widetilde{u}, v\rangle_{L_2(\Gamma)} - \langle v,\lambda-\widetilde{\lambda}\rangle_{L_2(\Gamma)} &= \langle Nf - \widetilde{N}f, v\rangle_{L_2(\Gamma)} \\ \langle u-\widetilde{u},\mu\rangle_{L_2(\Gamma_D)} &= 0 \end{aligned}$$

for all $(v,\mu) \in H^{1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma_D)$. Defining $u_E := u-\widetilde{u}$ and $\lambda_E := \lambda-\widetilde{\lambda}$ we get that $(u_E,\lambda_E) \in H^{1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma_D)$ satisfies the saddle point problem

$$\begin{aligned} \langle \widetilde{S}u_E, v\rangle_{L_2(\Gamma)} - \langle v,\lambda_E\rangle_{L_2(\Gamma)} &= \langle (\widetilde{S}-S)u, v\rangle_{L_2(\Gamma)} + \langle Nf - \widetilde{N}f, v\rangle_{L_2(\Gamma)} \\ \langle u_E,\mu\rangle_{L_2(\Gamma_D)} &= 0 \end{aligned}$$

for all $(v,\mu) \in H^{1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma_D)$. Applying Theorem 1.2 this gives

$$||u_E||_{H^{1/2}(\Gamma)} + ||\lambda_E||_{\widetilde{H}^{-1/2}(\Gamma_D)} \\ \le c \cdot \Big\{ ||(S-\widetilde{S})u||_{H^{-1/2}(\Gamma)} + ||(N-\widetilde{N})f||_{H^{-1/2}(\Gamma)} \Big\}. \quad \square$$

Let

$$\begin{aligned} X_h &= \operatorname{span}\{\varphi_k\}_{k=1}^M \subset X = H^{1/2}(\Gamma), \\ \Pi_h &= \operatorname{span}\{\chi_\ell\}_{\ell=1}^N \subset \Pi = \widetilde{H}^{-1/2}(\Gamma_D) \end{aligned}$$

denote a pair of conforming trial spaces satisfying the discrete inf–sup condition,

$$\inf_{0\neq\mu_h\in\Pi_h} \sup_{0\neq v_h\in X_h} \frac{b(v_h,\mu_h)}{||v_h||_X ||\mu_h||_\Pi} \ge \widetilde{\gamma}_S > 0\,. \tag{4.25}$$

To ensure the inf–sup condition 4.25 by Theorem 1.4 we need to have a bounded projection operator $P_h : X \to X_h$ satisfying

$$\langle P_h u, \mu_h \rangle_{L_2(\Gamma)} \;=\; \langle u, \mu_h \rangle_{L_2(\Gamma)} \quad \text{for all } \mu_h \in \Pi_h. \tag{4.26}$$

Since $\Pi_h \subset \widetilde{H}^{-1/2}(\Gamma_D)$ we assume assume that $\Pi_h \subset \widetilde{\Pi}_h$ where

$$\widetilde{\Pi}_h = \operatorname{span}\{\chi_\ell\}_{\ell=1}^{\widetilde{N}} \subset H^{-1/2}(\Gamma)$$

is a trial space defined on Γ. Then we can chose $P_h := \widetilde{Q}_h : X \to X_h$ defined by

$$\langle \widetilde{Q}_h u, \mu_h \rangle_{L_2(\Gamma)} \;=\; \langle u, \mu_h \rangle_{L_2(\Gamma)} \quad \text{for all } \mu_h \in \widetilde{\Pi}_h. \tag{4.27}$$

Note that we also need to have the stability estimate

$$||\widetilde{Q}_h v||_{H^{1/2}(\Gamma)} \;\le\; ||v||_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma) \tag{4.28}$$

to apply Theorem 1.4. Hence we have to define the trial spaces X_h and Π_h as discussed in Chapter 2.

The Galerkin variational problem of (4.21) is: find $(\widetilde{u}_h, \widetilde{\lambda}_h) \in X_h \times \Pi_h$ such that

$$\begin{aligned} \langle \widetilde{S}\widetilde{u}_h, v_h \rangle_{L_2(\Gamma)} - \langle v_h, \widetilde{\lambda}_h \rangle_{L_2(\Gamma_D)} &= \langle v_h, g_N \rangle_{L_2(\Gamma_N)} + \langle \widetilde{N} f, v_h \rangle_{L_2(\Gamma)} \\ \langle \widetilde{u}_h, \mu_h \rangle_{L_2(\Gamma_D)} &= \langle g_D, \mu_h \rangle_{L_2(\Gamma_D)} \end{aligned} \tag{4.29}$$

for all $(v_h, \mu_h) \in X_h \times \Pi_h$.

Applying Theorem 1.3 there exists a unique solution $(\widetilde{u}_h, \widetilde{\lambda}_h) \in X_h \times \Pi_h$ of (4.29) satisfying the following error estimate:

Theorem 4.7. *Let $(u, \lambda) \in H^{1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma_D)$ be the exact Cauchy data of (3.1) Then there hold the error estimates for the solution $(\widetilde{u}_h, \widetilde{\lambda}_h)$ of (4.29),*

$$\begin{aligned} ||u - \widetilde{u}_h||_{H^{1/2}(\Gamma)} \le c \cdot \Big\{ &\inf_{v_h \in X_h} ||u - v_h||_{H^{1/2}(\Gamma)} \\ &+ ||(S - \widetilde{S})u||_{H^{-1/2}(\Gamma)} + ||(N - \widetilde{N})f||_{H^{-1/2}(\Gamma)} \Big\} \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} &||\lambda - \widetilde{\lambda}_h||_{\widetilde{H}^{-1/2}(\Gamma_D)} \\ &\quad \le c_1 \cdot \left\{ \inf_{v_h \in X_h} ||u - v_h||_{H^{1/2}(\Gamma)} + \inf_{\mu_h \in \Pi_h} ||\lambda - \mu_h||_{\widetilde{H}^{-1/2}(\Gamma_D)} \right\} \\ &\quad + c_2 \cdot \left\{ ||(S - \widetilde{S})u||_{H^{-1/2}(\Gamma)} + ||(N - \widetilde{N})f||_{H^{-1/2}(\Gamma)} \right\}. \end{aligned} \tag{4.31}$$

Proof. Applying Theorem 1.3 we get

$$||\widetilde{u} - \widetilde{u}_h||_{H^{1/2}(\Gamma)} \le c_1 \cdot \inf_{v_h \in X_h} ||\widetilde{u} - v_h||_{H^{1/2}(\Gamma)},$$

$$\begin{aligned} ||\widetilde{\lambda} - \widetilde{\lambda}_h||_{\widetilde{H}^{-1/2}(\Gamma_D)} \le c_2 \cdot \Big\{ &\inf_{v_h \in X_h} ||\widetilde{u} - v_h||_{H^{1/2}(\Gamma)} \\ &+ \inf_{\mu_h \in \Pi_h} ||\widetilde{\lambda} - \mu_h||_{\widetilde{H}^{-1/2}(\Gamma_D)} \Big\}. \end{aligned}$$

Using

$$\inf_{v_h \in X_h} \|\widetilde{u} - v_h\|_{H^{1/2}(\Gamma)} \leq \|u - \widetilde{u}\|_{H^{1/2}(\Gamma)} + \inf_{v_h \in X_h} \|u - v_h\|_{H^{1/2}(\Gamma)},$$

$$\inf_{\mu_h \in \Pi_h} \|\widetilde{\lambda} - \mu_h\|_{\widetilde{H}^{-1/2}(\Gamma_D)} \leq \|\lambda - \widetilde{\lambda}\|_{\widetilde{H}^{-1/2}(\Gamma_D)} + \inf_{\mu_h \in \Pi_h} \|\lambda - \mu_h\|_{\widetilde{H}^{-1/2}(\Gamma_D)}$$

and (4.24) the error estimates follow by applying some triangle inequalities. □

To this end we will reconsider the saddle point problem (4.21). Using the approximate Dirichlet–Neumann map (3.56) which is defined via the finite element approximation (3.74)–(3.75), we get from (4.21) the variational problem: find $\widetilde{u} \in H^{1/2}(\Gamma)$, $\lambda \in \widetilde{H}^{-1/2}(\Gamma_D)$, $u_0 \in H^1_0(\Omega, \Gamma)$ such that

$$\begin{aligned} a(u_0 + \mathcal{E}\widetilde{u}, \mathcal{E}w) - b(w, \lambda) &= \langle w, g_N \rangle_{L_2(\Gamma_N)} + \langle f, \mathcal{E}w \rangle_{L_2(\Omega)} \\ a(u_0 + \mathcal{E}\widetilde{u}, v) &= \langle f, v \rangle_{L_2(\Omega)} \\ b(\widetilde{u}, \mu) &= \langle g_D, \mu \rangle_{L_2(\Gamma)} \end{aligned} \tag{4.32}$$

for all $w \in H^{1/2}(\Gamma)$, $\mu \in \widetilde{H}^{-1/2}(\Gamma_D)$ and $v \in H^1_0(\Omega)$. Since $\hat{u} := u_0 + \mathcal{E}\widetilde{u} \in H^1(\Omega)$, this is equivalent to find $\hat{u} \in H^1(\Omega)$ and $\lambda \in \widetilde{H}^{-1/2}(\Gamma_D)$ such that

$$\begin{aligned} a(\hat{u}, v) - b(v, \lambda) &= \langle f, v \rangle_{L_2(\Omega)} + \langle g_N, v \rangle_{L_2(\Gamma_N)} \\ b(\hat{u}, \mu) &= \langle g_D, \mu \rangle_{L_2(\Gamma_D)} \end{aligned} \tag{4.33}$$

for all $v \in H^1(\Omega)$ and $\mu \in \widetilde{H}^{-1/2}(\Gamma_D)$. Note that (4.33) is nothing else than the standard variational formulation with Lagrange parameter as introduced in [4, 15]. Note that in the discrete case the saddle point problem (4.33) can be obtained only when using trial spaces X_h and $\widetilde{X}_h$ satisfying the matching condition $X_h = \widetilde{X}_{h|\Gamma}$.

To describe the linear system which is equivalent to the Galerkin saddle point problem (4.29) we can use the stiffness matrix $\widetilde{S}_h$ as given in (4.17) and (4.20). In addition, we define a matrix B_h by

$$B_h[k, \ell] = \langle \varphi_k, \chi_\ell \rangle_{L_2(\Gamma_D)} \quad \text{for } k = 1, \ldots, M; \ell = 1, \ldots, N.$$

Then, (4.29) is equivalent to

$$\begin{pmatrix} \widetilde{S}_h & -B_h^\top \\ B_h & 0 \end{pmatrix} \begin{pmatrix} \underline{\widetilde{u}} \\ \underline{\widetilde{\lambda}} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}. \tag{4.34}$$

Using the finite element approximation (4.17) the linear system (4.34) can be written as coupled system,

$$\begin{pmatrix} A_{h,\Omega,\Omega} & A_{h,\Gamma,\Omega} & 0 \\ A_{h,\Gamma,\Omega}^\top & A_{h,\Gamma,\Gamma} & -B_h^\top \\ 0 & B_h & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_0 \\ \underline{\widetilde{u}} \\ \underline{\widetilde{\lambda}} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \\ \underline{g} \end{pmatrix}. \tag{4.35}$$

Defining

$$A_h := \begin{pmatrix} A_{h,\Omega,\Omega} & A_{h,\Gamma,\Omega} \\ A_{h,\Gamma,\Omega}^\top & A_{h,\Gamma,\Gamma} \end{pmatrix}, \ \widetilde{B}_h^\top := \begin{pmatrix} 0 \\ B_h^\top \end{pmatrix}, \ \underline{u} := \begin{pmatrix} \underline{u}_0 \\ \underline{\widetilde{u}} \end{pmatrix}, \ \underline{\hat{f}} := \begin{pmatrix} \underline{0} \\ \underline{f} \end{pmatrix},$$

(4.35) is the same as

$$\begin{pmatrix} A_h & -\widetilde{B}_h^\top \\ \widetilde{B}_h & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{\widetilde{\lambda}} \end{pmatrix} = \begin{pmatrix} \underline{\hat{f}} \\ \underline{g} \end{pmatrix}. \tag{4.36}$$

Note that (4.36) is the discrete system of the standard saddle point formulation (4.33).

Finally we consider the boundary element representation (4.20) for the discrete approximate Steklov–Poincaré operator $\widetilde{S}_h$. Then, (4.34) is equivalent to

$$\begin{pmatrix} D_h + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h) & -B_h^\top \\ B_h & 0 \end{pmatrix} \begin{pmatrix} \underline{\widetilde{u}} \\ \underline{\widetilde{\lambda}} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix} \tag{4.37}$$

as well as to

$$\begin{pmatrix} V_h & -\frac{1}{2}M_h - K_h & 0 \\ \frac{1}{2}M_h^\top + K_h^\top & D_h & -B_h^\top \\ 0 & B_h & 0 \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{\widetilde{u}} \\ \underline{\widetilde{\lambda}} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \\ \underline{g} \end{pmatrix}. \tag{4.38}$$

Note that (4.38) can be seen as Galerkin discretization of a two–fold saddle point problem with an additional term defined by the hypersingular integral operator.

5

Hybrid Coupled Domain Decomposition Methods

In this chapter we consider the mixed boundary value problem (3.1) when the domain Ω is given by a domain decomposition into p non–overlapping subdomains,

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j. \tag{5.1}$$

We assume that the subdomain boundaries $\Gamma_i := \partial\Omega_i$ are Lipschitz each. For neighbored subdomains Ω_i and Ω_j sharing either a common edge ($n = 2$) or a common face ($n = 3$) we define local coupling boundaries as

$$\overline{\Gamma}_{ij} := \Gamma_i \cap \Gamma_j \quad \text{for } i < j. \tag{5.2}$$

The skeleton of the domain decomposition is given by

$$\Gamma_S := \bigcup_{i=1}^{p} \Gamma_i = \Gamma \cup \bigcup_{i<j} \overline{\Gamma}_{ij}\,. \tag{5.3}$$

Instead of the mixed boundary value problem (3.1) we now consider local subproblems to find weak solutions $u_i \in H^1(\Omega_i)$ satisfying

$$\begin{aligned} L_i(x)u_i(x) &= f(x) && \text{for } x \in \Omega_i, \\ \gamma_0^i u_i(x) &= g_D(x) && \text{for } x \in \Gamma_D \cap \Gamma_i, \\ \gamma_1^i u_i(x) &= g_N(x) && \text{for } x \in \Gamma_N \cap \Gamma_i. \end{aligned} \tag{5.4}$$

In addition to the boundary conditions in (5.4), the local Cauchy data $(\gamma_0^i u_i, \gamma_1^i u_i)$ have to satisfy some transmission conditions on the local coupling boundaries Γ_{ij},

$$\gamma_0^i u_i(x) = \gamma_0^j u_j(x), \quad \gamma_1^i u_i(x) + \gamma_1^j u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{5.5}$$

It is worth to mention that the local subproblems (5.4) are well defined and can be solved locally, when the Dirichlet data $\gamma_0^i u_i$ are given on the skeleton Γ_S.

O. Steinbach: LNM 1809, pp. 85–115, 2003.

This observation motivates the following approach to determine $u \in H^{1/2}(\Gamma_S)$ satisfying the Dirichlet boundary conditions $u(x) = g_D(x)$ for $x \in \Gamma_D$ as well as the transmission conditions (5.5). Here, $H^{1/2}(\Gamma_S)$ is a space of functions defined on the skeleton Γ_S equipped with the norm

$$||v||_{H^{1/2}(\Gamma_S)} := \left\{ \sum_{i=1}^{p} ||v_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)} \right\}^{1/2} . \tag{5.6}$$

Moreover,

$$H_0^{1/2}(\Gamma_S, \Gamma_D) := \Big\{ v \in H^{1/2}(\Gamma_S) \, : \, v(x) = 0 \quad \text{for } x \in \Gamma_D \Big\} . \tag{5.7}$$

For given $u \in H^{1/2}(\Gamma_S)$ we can consider local Dirichlet problems

$$L_i(x)u_i(x) = f(x) \quad \text{for } x \in \Omega_i, \quad \gamma_0^i u_i(x) = u(x) \quad \text{for } x \in \Gamma_i \tag{5.8}$$

and the Dirichlet transmission conditions in (5.5) are trivially satisfied. By solving the local Dirichlet boundary value problems (5.8), in particular applying the Dirichlet–Neumann map (3.16) locally, the associated Neumann data are given by

$$\lambda_i(x) := S_i u_{|\Gamma_i}(x) - N_i f(x) \quad \text{for } x \in \Gamma_i \tag{5.9}$$

which have to satisfy both the Neumann boundary conditions and the Neumann transmission conditions,

$$\lambda_i(x) = g_N(x) \quad \text{for } x \in \Gamma_i \cap \Gamma_N, \quad \lambda_i(x) + \lambda_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{5.10}$$

To derive variational formulations of the local subproblems (5.4) coupled by the transmission conditions (5.5) we consider the local Dirichlet–Neumann maps (5.9) together with either a weak formulation of (5.10) or a variational formulation of (5.8) with Lagrange multipliers. However, for a practical realization we have to replace the local Steklov–Poincaré operators S_i and the Newton potentials $N_i f$ in (5.9) by suitable approximations $\widetilde{S}_i$ and $\widetilde{N}_i f$ as introduced in Chapter 3. Hence we consider a coupled problem to find $\widetilde{u} \in H^{1/2}(\Gamma_S)$ with $\widetilde{u}(x) = g_D(x)$ for $x \in \Gamma_D$, $\widetilde{u}_i \in H^{1/2}(\Gamma_i)$ and $\widetilde{\lambda}_i \in H^{-1/2}(\Gamma_i)$ such that

$$\begin{aligned}
\widetilde{\lambda}_i(x) &= \widetilde{S}_i \widetilde{u}_i(x) - \widetilde{N}_i f(x) && \text{for } x \in \Gamma_i \\
\widetilde{u}_i(x) &= \widetilde{u}_{|\Gamma_i}(x) && \text{for } x \in \Gamma_i, \\
\widetilde{\lambda}_i(x) &= g_N(x) && \text{for } x \in \Gamma_i \cap \Gamma_N, \\
0 &= \widetilde{\lambda}_i(x) + \widetilde{\lambda}_j(x) && \text{for } x \in \Gamma_{ij}.
\end{aligned} \tag{5.11}$$

The global boundary value problem (5.4) and (5.5) is therefore reduced to coupled Steklov–Poincaré operator equations on the skeleton of the domain decomposition. The approximate Steklov–Poincaré operators are defined locally using suitable trial spaces, i.e. finite or boundary elements. For a global discretization of the coupled problem (5.11), trial spaces for the Cauchy data on the skeleton have to be introduced. It is obvious, that the local approximations of the Steklov–Poincaré operators can be done independent of the global trial space on the skeleton.

5.1 Dirichlet Domain Decomposition Methods

We first consider variational formulations for domain decomposition methods based on a strong coupling of the Dirichlet data, in particular, for $\widetilde{u} \in H^{1/2}(\Gamma_S)$ we define

$$\widetilde{u}_i(x) := \widetilde{u}(x) \quad \text{for } x \in \Gamma_i. \tag{5.12}$$

Then, the Dirichlet transmission conditions in (5.11) are trivially satisfied. For the Neumann boundary and Neumann transmission conditions we consider a weak coupling, in particular we require

$$\int_{\Gamma_{ij}} [\widetilde{\lambda}_i(x) + \widetilde{\lambda}_j(x)] v_{ij}(x) ds_x = 0 \quad \text{for all } v_{ij} \in H^{1/2}(\Gamma_{ij})$$

and

$$\int_{\Gamma_i \cap \Gamma_N} [\widetilde{\lambda}_i(x) - g_N(x)] v_N(x) ds_x = 0 \quad \text{for all } v_N \in H^{1/2}(\Gamma_N)\,.$$

Taking the sum over a local subdomain boundaries Γ_{ij} and the Neumann boundary Γ_N gives

$$\sum_{i=1}^{p} \int_{\Gamma_i} \widetilde{\lambda}_i(x) v(x) ds_x = \int_{\Gamma_N} g_N(x) v(x) ds_x \tag{5.13}$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma_D)$. Using the local Dirichlet–Neumann maps in (5.11) we get the variational problem: find $\widetilde{u} \in H^{1/2}(\Gamma_S)$ satisfying $\widetilde{u}(x) = g_D(x)$ for $x \in \Gamma_D$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \widetilde{S}_i \widetilde{u}_{|\Gamma_i}(x) v_{|\Gamma_i}(x) ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} \widetilde{N}_i f(x) v_{|\Gamma_i}(x) ds_x + \int_{\Gamma_N} g_N(x) v_{|\Gamma_N}(x) ds_x \tag{5.14}$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma_D)$.

Theorem 5.1. [45] *The bilinear form*

$$a(u,v) := \sum_{i=1}^{p} \int_{\Gamma_i} \widetilde{S}_i u_{|\Gamma_i}(x) v_{|\Gamma_i}(x) ds_x \tag{5.15}$$

is bounded,

$$|a(u,v)| \leq c_2^{\widetilde{S}} \cdot ||u||_{H^{1/2}(\Gamma_S)} ||v||_{H^{1/2}(\Gamma_S)} \quad \text{for all } u,v \in H^{1/2}(\Gamma_S) \tag{5.16}$$

and elliptic on $H_0^{1/2}(\Gamma_S, \Gamma_D)$,

$$a(v,v) \geq c_1^{\widetilde{S}} \cdot ||v||^2_{H^{1/2}(\Gamma_S)} \quad \text{for all } v \in H_0^{1/2}(\Gamma_S,\Gamma_D). \tag{5.17}$$

The positive constants are given by $c_1^{\widetilde{S}} := \min_{i=1,\dots,p} c_1^{\widetilde{S}_i}$, $c_2^{\widetilde{S}} := \max_{i=1,\dots,p} c_2^{\widetilde{S}_i}$.

Proof. Since all local approximate Steklov–Poincaré operators are bounded, see Theorem 3.5 for a finite element approximation and Theorem 3.7 when using boundary elements, we get

$$\begin{aligned}
|a(u,v)| &\leq \sum_{i=1}^p |\langle \widetilde{S}_i u_{|\Gamma_i}, v_{|\Gamma_i}\rangle_{L_2(\Gamma_i)}| \leq \sum_{i=1}^p c_2^{\widetilde{S}_i} ||u_{|\Gamma_i}||_{H^{1/2}(\Gamma_i)} ||v_{|\Gamma_i}||_{H^{1/2}(\Gamma_i)} \\
&\leq \max_{i=1,\dots,p} c_2^{\widetilde{S}_i} \cdot \left(\sum_{i=1}^p ||u_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)} \right)^{1/2} \left(\sum_{i=1}^p ||v_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)} \right)^{1/2} \\
&= \max_{i=1,\dots,p} c_2^{\widetilde{S}_i} \cdot ||u||_{H^{1/2}(\Gamma_S)} ||v||_{H^{1/2}(\Gamma_S)} \qquad \text{for all } u,v \in H^{1/2}(\Gamma_S).
\end{aligned}$$

For $u \in H_0^{1/2}(\Gamma_S,\Gamma_D)$ we have $u(x) = 0$ for $x \in \Gamma_D$. Since there is at least one subdomain boundary Γ_{i^*} with $\Gamma_{i^*} \cap \Gamma_D \neq \emptyset$ we conclude $u_{|\Gamma_{i^*}} \in H^{1/2}(\Gamma_{i^*})_{/\mathcal{R}_i}$. We can repeat this argument recursively to get $u_{|\Gamma_i} \in H^{1/2}(\Gamma_i)_{/\mathcal{R}_i}$ for all $i = 1,\dots,p$. Hence we have, using the symmetric representation (3.40),

$$\langle \widetilde{S}_i u_{|\Gamma_i}, u_{|\Gamma_i}\rangle_{L_2(\Gamma_i)} \geq \langle D_i u_{|\Gamma_i}, u_{|\Gamma_i}\rangle_{L_2(\Gamma_i)} \geq c_1^{\widetilde{S}_i} ||u_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)}$$

Summation over $i = 1,\dots,p$ gives (5.17). □

Therefore we have unique solvability of (5.14) due to the Lax–Milgram theorem.

Theorem 5.2. *Let* $u \in H^{1/2}(\Gamma_S)$ *be the unique solution of the coupled problem (5.9)–(5.10) and let* $\widetilde{u} \in H^{1/2}(\Gamma_S)$ *be the unique solution of (5.14), respectively. Then,*

$$||u-\widetilde{u}||^2_{H^{1/2}(\Gamma_S)} \leq \frac{2}{[c_1^{\widetilde{S}}]^2} \sum_{i=1}^p \left[||(S_i - \widetilde{S}_i)u||^2_{H^{-1/2}(\Gamma_i)} + ||(N_i - \widetilde{N}_i)f||^2_{H^{-1/2}(\Gamma_i)} \right]. \tag{5.18}$$

Proof. First we note that $u \in H^{1/2}(\Gamma_S)$ with $u(x) = g_D(x)$ for $x \in \Gamma_D$ satisfies

$$\sum_{i=1}^p \int_{\Gamma_i} S_i u_{|\Gamma_i}(x) v_{|\Gamma_i}(x) ds_x = \sum_{i=1}^p \int_{\Gamma_i} N_i f(x) v_{|\Gamma_i}(x) ds_x + \int_{\Gamma_N} g_N(x) v_{|\Gamma_N}(x) ds_x$$

for all $v \in H_0^{1/2}(\varGamma_S, \varGamma_D)$. Subtracting this from (5.14) we get the variational equality

$$\sum_{i=1}^{p} \langle S_i u - \widetilde{S}_i \widetilde{u}, v \rangle_{L_2(\varGamma_i)} = \sum_{i=1}^{p} \langle N_i f - \widetilde{N}_i f, v \rangle_{L_2(\varGamma_i)} \quad \text{for all } v \in H_0^{1/2}(\varGamma_S, \varGamma_D)\,.$$

Using (5.17) for $v := u - \widetilde{u} \in H_0^{1/2}(\varGamma_S, \varGamma_D)$,

$$\begin{aligned}
c_1^{\widetilde{S}} \cdot ||u - \widetilde{u}||^2_{H^{1/2}(\varGamma_S)} &\leq \sum_{i=1}^{p} \langle \widetilde{S}_i (u - \widetilde{u}), u - \widetilde{u} \rangle_{L_2(\varGamma_i)} \\
&= \sum_{i=1}^{p} \Big[\langle (\widetilde{S}_i - S_i) u, u - \widetilde{u} \rangle_{L_2(\varGamma_i)} + \langle (N_i - \widetilde{N}_i) f, u - \widetilde{u} \rangle_{L_2(\varGamma_i)} \Big] \\
&\leq \sum_{i=1}^{p} \Big[||(S_i - \widetilde{S}_i) u||_{H^{-1/2}(\varGamma_i)} + ||(N_i - \widetilde{N}_i) f||_{H^{-1/2}(\varGamma_i)} \Big] \, ||u - \widetilde{u}||_{H^{1/2}(\varGamma_i)} \\
&\leq \left\{ \sum_{i=1}^{p} \Big[||(S_i - \widetilde{S}_i) u||_{H^{-1/2}(\varGamma_i)} + ||(N_i - \widetilde{N}_i) f||_{H^{-1/2}(\varGamma_i)} \Big]^2 \right\}^{1/2} \\
&\qquad \cdot ||u - \widetilde{u}||_{H^{1/2}(\varGamma_S)}. \quad \Box
\end{aligned}$$

Let

$$X_h := \operatorname{span}\{\varphi_k\}_{k=1}^{M} \subset H_0^{1/2}(\varGamma_S, \varGamma_D) \tag{5.19}$$

be a global finite element trial space on the skeleton $\varGamma_S$. By restriction onto $\varGamma_i$ we also define local trial spaces

$$X_{h,i} := \operatorname{span}\{\varphi_k^i\}_{k=1}^{M_i}\,. \tag{5.20}$$

Obviously, for any $\varphi_k^i \in X_{h,i}$ there exists a unique basis function $\varphi_k \in X_h$ with $\varphi_k^i = \varphi_{k|\varGamma_i}$. By using the isomorphisms

$$\underline{v}_i \in \mathbb{R}^{M_i} \leftrightarrow v_{h,i} = \sum_{k=1}^{M_i} v_{i,k} \varphi_k^i \in X_{h,i}, \quad \underline{v} \in \mathbb{R}^M \leftrightarrow v_h = \sum_{k=1}^{M} v_k \varphi_k \in X_h,$$

there exist connectivity matrices $A_i \in \mathbb{R}^{M_i \times M}$ such that

$$\underline{v}_i = A_i \underline{v}\,. \tag{5.21}$$

Let $\widetilde{g}_D \in H^{1/2}(\varGamma_S)$ be an arbitrary but fixed extension of the given Dirichlet data with $\widetilde{g}_D(x) = g_D(x)$ for $x \in \varGamma_D$. Then we have to find $\widetilde{u}_0 \in H_0^{1/2}(\varGamma_S, \varGamma_D)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \widetilde{S}_i \widetilde{u}_{0|\Gamma_i}(x) v_{|\Gamma_i}(x) ds_x = \\ = \sum_{i=1}^{p} \int_{\Gamma_i} [\widetilde{N}_i f(x) - \widetilde{S}_i \widetilde{g}_D(x)] v_{|\Gamma_i}(x) ds_x + \int_{\Gamma_N} g_N(x) v_{|\Gamma_N}(x) ds_x \tag{5.22}$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma_D)$. The Galerkin variational formulation of (5.22) is: find $\widetilde{u}_{0,h} \in X_h$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \widetilde{S}_i \widetilde{u}_{0,h|\Gamma_i}(x) v_{h|\Gamma_i}(x) ds_x = \\ = \sum_{i=1}^{p} \int_{\Gamma_i} [\widetilde{N}_i f(x) - \widetilde{S}_i \widetilde{g}_D(x)] v_{h|\Gamma_i}(x) ds_x + \int_{\Gamma_N} g_N(x) v_{h|\Gamma_N}(x) ds_x \tag{5.23}$$

for all $v_h \in X_h$.

Note that (5.23) is uniquely solvable due to Theorem 5.1. The Galerkin approximation of $\widetilde{u}$ is then defined by $\widetilde{u}_h := \widetilde{u}_{0,h} + \widetilde{g}_D$. As in Theorem 4.2 we get the following error estimate.

Theorem 5.3. *Let $u \in H^{1/2}(\Gamma_S)$ be the unique solution of the coupled problem (5.9)–(5.10) and let $\widetilde{u}_{0,h} \in X_h$ be its Galerkin approximation. Then there holds the quasi–optimal error estimate*

$$||u - \widetilde{u}_h||^2_{H^{1/2}(\Gamma_S)} \le c_1 \cdot \inf_{v_h \in X_h} ||(u - \widetilde{g}_D) - v_h||^2_{H^{1/2}(\Gamma_S)} \\ + c_2 \cdot \sum_{i=1}^{p} ||(S_i - \widetilde{S}_i) u_{|\Gamma_i}||^2_{H^{-1/2}(\Gamma_i)}. \tag{5.24}$$

Based on the local trial spaces $X_{h,i}$ we can define local Galerkin matrices $\widetilde{S}_{h,i}$ of the approximated Steklov–Poincaré operators $\widetilde{S}_i$ as introduced in (4.17) using a finite element approximation or in (4.20) when using boundary elements. The global Galerkin stiffness matrix $\widetilde{S}_h$ is then given by assembling and we have to solve the linear system of equations,

$$\widetilde{S}_h \underline{u} = \sum_{i=1}^{p} A_i^\top \widetilde{S}_{h,i} A_i \underline{u} = \underline{f} \tag{5.25}$$

where the right hand side is given by the linear form in (5.23).

When using a boundary element approximation of the local Steklov–Poincaré operator S_i, we get as resulting error estimate,

$$||u - \widetilde{u}_h||^2_{H^{1/2}(\Gamma_S)} \le c_1 \cdot \inf_{v_h \in X_h} ||(u - \widetilde{g}_D) - v_h||^2_{H^{1/2}(\Gamma_S)} \\ + c_2 \cdot \sum_{i=1}^{p} \inf_{w_{h,i} \in Z_{h,i}} ||S_i u_{|\Gamma_i} - w_{h,i}||^2_{H^{-1/2}(\Gamma_i)}. \tag{5.26}$$

The local stiffness matrices are then given by (see (4.20)),

$$\widetilde{S}_{h,i} = D_{h,i} + (\frac{1}{2}M_{h,i}^\top + K_{h,i}^\top)V_{h,i}^{-1}(\frac{1}{2}M_{h,i} + K_{h,i}).$$

Inserting this into (5.25) gives

$$\sum_{i=1}^p A_i^\top [D_{h,i} + (\frac{1}{2}M_{h,i}^\top + K_{h,i}^\top)V_{h,i}^{-1}(\frac{1}{2}M_{h,i} + K_{h,i})]A_i\underline{u} = \underline{f} \tag{5.27}$$

which is the discrete counter part of the Galerkin problem (5.23). Defining

$$V_h := \operatorname{diag}(V_{h,i})_{i=1}^p, \quad \underline{w}_i := V_{h,i}^{-1}(\frac{1}{2}M_{h,i} + K_{h,i})A_i\underline{u}, \quad \underline{w} := (\underline{w}_i)_{i=1}^p$$

as well as the assembled matrices

$$D_h := \sum_{i=1}^p A_i^\top D_{h,i}A_i, \quad (\frac{1}{2}M_h + K_h) := \sum_{i=1}^p (\frac{1}{2}M_{h,i} + K_{h,i})A_i$$

we obtain a positive definite but block skew–symmetric system,

$$\begin{pmatrix} V_h & -\frac{1}{2}M_h - K_h \\ \frac{1}{2}M_h^\top + K_h^\top & D_h \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \end{pmatrix}. \tag{5.28}$$

Note that the local trial spaces $Z_{h,i} \subset H^{-1/2}(\Gamma_i)$ can be defined independently of the global trial space $X_h \subset H_0^{1/2}(\Gamma_S, \Gamma_0)$. For example, the global trial space X_h can be defined by using piecewise linear continuous basis functions, while the local trial spaces $Z_{h,i}$ are defined by piecewise constant basis functions using the global mesh locally. However, it seems to be favorable to use local trial spaces $Z_{h,i}$ which are completely independent of the global space X_h. This may lead to more efficient and more accurate algorithms. For preconditioned iterative methods to solve (5.28) efficiently, see for example [27, 59, 69]. Note that, due to Lemma 4.4, the local matrices $\widetilde{S}_{h,i}$ are spectrally equivalent to the exact Galerkin matrices $S_{h,i}$ independent of the choice of the local trial spaces $Z_{h,i} \subset H^{-1/2}(\Gamma_i)$.

Now we consider the case when using a finite element approximation of the local Steklov–Poincaré operators. Then we get as resulting error estimate,

$$\begin{aligned} ||u - \widetilde{u}_h||^2_{H^{1/2}(\Gamma_S)} \leq c_1 \cdot \inf_{v_h \in X_h} ||(u - \widetilde{g}_D) - v_h||^2_{H^{1/2}(\Gamma_S)} \\ + c_2 \cdot \sum_{i=1}^p \inf_{v_{h,i} \in \widetilde{X}_{h,i}} ||u - v_{h,i}||^2_{H^1(\Omega_i)}. \end{aligned} \tag{5.29}$$

Using the approximate Dirichlet–Neumann map (3.74) locally, i.e. replacing the approximations $\widetilde{S}_i$ and $\widetilde{N}_i f$ in (5.23) and using local bilinear forms $a_i(\cdot, \cdot)$

related to the local partial differential operators L_i, the Galerkin problem (5.23) reads: find $\widetilde{u}_{0,h} \in X_h$ such that

$$\sum_{i=1}^{p} a_i(\mathcal{E}_i\widetilde{u}_{0,h}+\mathcal{E}_i\widetilde{g}_D+u_{0,h,i},\mathcal{E}_i w_h) = \sum_{i=1}^{p}\int_{\Omega_i} f(x)\mathcal{E}_i w_h(x)dx+\int_{\Gamma_N} g_N(x)w_h(x)ds_x \tag{5.30}$$

for all $w_h \in X_h$ where $u_{0,h,i} \in \widetilde{X}_{h,i}$ are unique solutions of

$$a_i(u_{0,h,i} + \mathcal{E}_i\widetilde{u}_{0,h} + \mathcal{E}_i\widetilde{g}_D, v_{h,i}) = \int_{\Omega_i} f(x)v_{h,i}(x)dx \tag{5.31}$$

for all $v_{h,i} \in \widetilde{X}_{h,i}$ and $i = 1,\ldots,p$. Defining the local matrices

$$A_{h,\Omega_i,\Omega_i}[\ell,k] = a_i(\phi_k^i,\phi_\ell^i), \quad A_{h,\Gamma_i,\Omega_i}[\ell,j] = a_i(\mathcal{E}_i\varphi_j^i,\phi_\ell^i)$$

for $k,\ell = 1,\ldots,\widetilde{M}_i$ and $j = 1,\ldots,M_i$ as well as local right hand side vectors by

$$f_\ell^{L,i} = \int_{\Omega_i} f(x)\phi_\ell^i(x)dx - a_i(\mathcal{E}_i\widetilde{g}_D,\phi_\ell^i) \quad \text{for } \ell = 1,\ldots,\widetilde{M}_i$$

the local problems (5.31) can be written as

$$A_{h,\Omega_i,\Omega_i}\underline{u}_0^i + A_{h,\Gamma_i,\Omega_i}A_i\underline{\widetilde{u}}_0 = \underline{f}^{L,i} \quad \text{for } i = 1,\ldots,p. \tag{5.32}$$

For $k,\ell = 1,\ldots,\widetilde{M}_i$ we define

$$A_{h,\Gamma_i,\Gamma_i}[\ell,k] = a_i(\mathcal{E}_i\varphi_k^i,\mathcal{E}_i\varphi_\ell^i)$$

$$f_\ell^{S,i} = \int_{\Omega_i} f(x)\mathcal{E}_i\varphi_\ell^i(x)dx + \int_{\Gamma_N} g_N(x)\varphi_\ell^i(x)ds_x - a_i(\mathcal{E}_i\widetilde{g}_D,\mathcal{E}_i\varphi_\ell^i)$$

Then we can write (5.30) as

$$\sum_{i=1}^{p}\left[A_i^\top A_{h,\Gamma_i,\Gamma_i}A_i\underline{\widetilde{u}}_0 + A_i^\top A_{h,\Gamma_i,\Omega_i}^\top\underline{u}_0^i\right] = \sum_{i=1}^{p}A_i^\top\underline{f}^{S,i} \tag{5.33}$$

or

$$\sum_{i=1}^{p}A_i^\top\widetilde{S}_{h,i}^{\mathrm{FEM}}A_i\underline{\widetilde{u}}_0 = \sum_{i=1}^{p}A_i^\top\underline{f}^{S,i}.$$

Hence we have to solve a coupled linear system,

$$\begin{pmatrix} A_{h,\Omega,\Omega} & A_{h,\Gamma,\Omega} \\ A_{h,\Gamma,\Omega}^\top & A_{h,\Gamma,\Gamma}\end{pmatrix}\begin{pmatrix}\underline{u}_0 \\ \underline{\widetilde{u}}_0\end{pmatrix} = \begin{pmatrix}\underline{f}^L \\ \underline{f}^S\end{pmatrix}. \tag{5.34}$$

Note that

$$A_{h,\Omega,\Omega} = \operatorname{diag}(A_{h,\Omega_i,\Omega_i})_{i=1}^p .$$

Hence we can compute $A^{-1}_{h,\Omega,\Omega}$ in parallel to get the Schur complement system

$$\widetilde{S}_h^{\text{FEM}}\widetilde{\underline{u}}_0 = \left[A_{h,\Gamma,\Gamma} - A^\top_{h,\Gamma,\Omega}A^{-1}_{h,\Omega,\Omega}A_{h,\Omega}\right]\widetilde{\underline{u}}_0 = \underline{f}^S - A^\top_{h,\Gamma,\Omega}A^{-1}_{h,\Omega,\Omega}\underline{f}^L . \tag{5.35}$$

Note that (5.35) is the linear system associated with the Galerkin formulation (5.23). The coupled linear system (5.34) corresponds to the standard system in finite element domain decomposition methods, see for example [17, 41]. In particular, the stiffness matrix in (5.34) is symmetric and positive definite. Hence we can use the techniques developed in [17, 41] for an efficient solution of (5.34) by a preconditioned conjugate gradient method in parallel. However, in contrast to the standard approach, the degrees of freedom $\widetilde{\underline{u}}_0$ on the skeleton are independent of the local degrees of freedom, $\underline{u}_0$, within the subdomains. Note that the local Schur complements $\widetilde{S}^{\text{FEM}}_{h,i}$ are spectrally equivalent to the exact Galerkin matrices $S_{h,i}$ of the local Steklov–Poincaré operators (see Lemma 4.3), and hence the same is true for the global Schur complements $\widetilde{S}^{\text{FEM}}_h$ and S_h. However, the coupled linear system (5.34) depend on the local trial spaces $\widetilde{X}_{h,i}$ (to solve the local Dirichlet problem) and $\widetilde{X}^*_{h,i}$ (to define the discrete harmonic extension). We will come back to this problem in the next section. There we describe an equivalent finite element domain decomposition method which is based on the standard approach. Introducing a two–level finite element space globally we can formulate a conforming method which allows the use of non–matching grids locally. In particular, we describe how to deal with the discrete extension operators used in the proposed approach.

Note that we may use a boundary element approximation of the local Steklov–Poincaré operators in a couple of subdomains Ω_i, $i = 1, \ldots, q < p$, while for the remaining subdomains Ω_i, $i = q+1, \ldots, p$, we use local approximations by finite elements. The resulting linear system is then given by assembling the discrete Steklov–Poincaré operators as given in (5.27) and (5.35), respectively:

$$\left[\sum_{i=1}^q A_i^\top \widetilde{S}^{BEM}_{h,i} A_i + \sum_{i=q+1}^p A_i^\top \widetilde{S}^{FEM}_{h,i} A_i\right]\widetilde{\underline{u}}_0 = \sum_{i=1}^q A_i^\top \underline{f}_i^{BEM} + \sum_{i=q+1}^p A_i^\top \underline{f}_i^{FEM}. \tag{5.36}$$

In analogy to (5.28) and (5.34) we also consider the coupled system

$$\begin{pmatrix} V_h & -\frac{1}{2}M_h - K_h & 0 \\ \frac{1}{2}M_h^\top + K_h^\top & D_h + A_{h,\Gamma,\Gamma} & A^\top_{h,\Gamma,\Omega} \\ 0 & A_{h,\Gamma,\Omega} & A_{h,\Omega,\Omega} \end{pmatrix} \begin{pmatrix} \underline{w} \\ \widetilde{\underline{u}}_0 \\ \underline{u}_0 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f}^S \\ \underline{f}^L \end{pmatrix} . \tag{5.37}$$

For efficient preconditioned iterative solution methods to solve (5.37), see [51]. Note that (5.37) can be seen as Galerkin discretization of a variational formulation which results from a symmetric coupling of finite and boundary element methods as introduced in [31].

5.2 A Two–Level Method

In this section we describe a two–level Galerkin discretization of (5.14) when using finite elements. The coarse grid space introduced here corresponds to the global trial space X_h used in (5.23) but the extension to the local subdomains is already included in the definition of X_h.

As a model problem we consider the Dirichlet boundary value problem

$$-\mathrm{div}\,\alpha(x)\nabla u(x) \;=\; f(x) \quad \text{for } x \in \Omega, \quad \gamma_0 u(x) = 0 \quad \text{for } x \in \Gamma \qquad (5.38)$$

where the domain Ω is given by the non–overlapping domain decomposition (5.1). We assume $\alpha(x) \geq \alpha_0 > 0$ for $x \in \Omega$. The variational formulation of (5.38) is: find $u \in H_0^1(\Omega)$ such that

$$\sum_{i=1}^{p} \int_{\Omega_i} \alpha(x)\nabla u(x)\nabla v(x)dx \;=\; \int_{\Omega} f(x)v(x)dx \quad \text{for all } v \in H_0^1(\Omega). \qquad (5.39)$$

Let $X_h \subset H_0^1(\Omega)$ be a global finite element trial space, then the Galerkin formulation of (5.39) is: find $u_h \in X_h$ such that

$$\sum_{i=1}^{p} \int_{\Omega_i} \alpha(x)\nabla u_h(x)\nabla v_h(x)dx \;=\; \int_{\Omega} f(x)v_h(x)dx \quad \text{for all } v_h \in X_h. \qquad (5.40)$$

Applying standard arguments, in particular Cea's lemma, we get unique solvability of (5.40) and the quasi–optimal error estimate

$$||u - u_h||_{H^1(\Omega)} \;\leq\; c \cdot \inf_{v_h \in X_h} ||u - v_h||_{H^1(\Omega)}.$$

In what follows we will construct a two–level trial space X_h which is suitable for the domain decomposition given by (5.1) and which allows the use of non–matching grids locally. We first define a coarse grid space of piecewise polynomial basis functions,

$$X_H \;:=\; \mathrm{span}\{\varphi_k\}_{k=1}^{M_0} \subset H_0^1(\Omega). \qquad (5.41)$$

Then we consider a decomposition of X_H given by

$$X_H \;:=\; X_{H,S} + \sum_{i=1}^{p} X_{H,i} \qquad (5.42)$$

with

$$X_{H,S} := \text{span}\{\varphi_k^S\}_{k=1}^{M_S}, \quad X_{H,i} := X_H \cap H_0^1(\Omega_i) = \text{span}\{\varphi_k^i\}_{k=1}^{M_i}. \quad (5.43)$$

For every subdomain Ω_i, $i = 1, \ldots, p$, we then define local trial spaces

$$X_{h,i} := \text{span}\{\phi_k^i\}_{k=1}^{\widetilde{M}_i} \subset H_0^1(\Omega_i) \quad (5.44)$$

where we assume that $X_{H,i} \cup X_{h,i}$ defines a stable basis locally. In particular, we need to ensure that the local stiffness matrices $A_{L,L}$ are invertible, see (5.48). In general we have to distinguish two cases, see Figure 5.1 and Figure 5.2 in case of piecewise linear basis functions. There, ■ denotes the global coarse grid nodes of X_H and • denotes the local fine grid nodes of $X_{h,i}$. In Figure 5.1 we show the case, that the local coarse grid space is nested in the local one, therefore $X_{H,i}$ and $X_{h,i}$ define a nested hierarchical two–level trial space locally, which is stable by construction. In a more general case as depicted in Figure 5.2 the local coarse grid space $X_{H,i}$ is non–nested in the fine grid space $X_{h,i}$. This may lead to an unstable basis, in particular, when a coarse grid function can be represented approximately by the fine grid basis. In this situation one may change either one of the trial spaces $X_{H,i}$ or $X_{h,i}$ to get nested spaces, or one has to refine the coarse grid space $X_{H,i}$ simultaneous to the fine grid space. Note that the definition of $X_{h,i}$, if $X_{H,i}$ is given, does not influence the properties of the Schur complement matrix $\widetilde{S}_h^{\text{FEM}}$ (see Lemma 4.3), but we need to have unique solvability of the local subproblems.

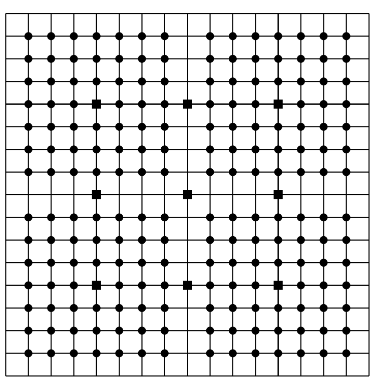


Figure 5.1:
Nested coarse grid space

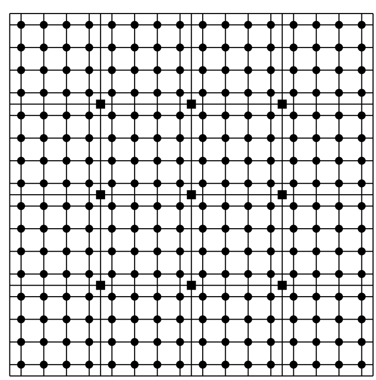


Figure 5.2:
Non–nested coarse grid space

Then,

$$X_h := X_{H,S} + \sum_{i=1}^{p} [X_{H,i} + X_{h,i}] \subset H_0^1(\Omega) \quad (5.45)$$

is the finite element space to be used in (5.40). According to (5.45) we make the trial

$$u_h := u_H + \sum_{i=1}^{p} [u_{H,i} + u_{h,i}] \in X_h$$

with

$$u_H \in X_{H,S}, \quad u_{H,i} \in X_{H,i}, \quad u_{h,i} \in X_{h,i} \ (i = 1, \ldots, p).$$

We define local matrices

$$A_{h,h,i}[\widetilde{\ell}, \widetilde{k}] = \int_{\Omega_i} \alpha(x) \nabla \phi^i_{\widetilde{k}}(x) \nabla \phi^i_{\widetilde{\ell}}(x) dx \qquad \text{for } \widetilde{k}, \widetilde{\ell} = 1, \ldots, \widetilde{M}_i,$$

$$A_{H,H,i}[\ell, k] = \int_{\Omega_i} \alpha(x) \nabla \varphi^i_k(x) \nabla \varphi^i_\ell(x) dx \qquad \text{for } k, \ell = 1, \ldots, M_i$$

$$A_{H,h,i}[\widetilde{\ell}, k] = \int_{\Omega_i} \alpha(x) \nabla \varphi^i_k(x) \nabla \phi^i_{\widetilde{\ell}}(x) dx \qquad \text{for } k = 1, \ldots, M_i; \ell = 1, \ldots, \widetilde{M}_i$$

as well as for $k = 1, \ldots, M_S$

$$A_{S,S}[\ell, k] = \sum_{i=1}^{p} \int_{\Omega_i} \alpha(x) \nabla \varphi^S_k(x) \nabla \varphi^S_\ell(x) dx \qquad \text{for } \ell = 1, \ldots, M_S;$$

$$A_{S,H,i}[j, k] = \int_{\Omega_i} \alpha(x) \nabla \varphi^S_k(x) \nabla \varphi^i_j(x) dx \qquad \text{for } j = 1, \ldots, M_i;$$

$$A_{S,h,i}[\widetilde{\ell}, k] = \int_{\Omega_i} \alpha(x) \nabla \varphi^S_k(x) \nabla \phi^i_{\widetilde{\ell}}(x) dx \qquad \text{for } \widetilde{\ell} = 1, \ldots, \widetilde{M}_i.$$

The variational problem (5.40) is then equivalent to the system of linear equations,

$$\begin{pmatrix} A_{h,h} & A_{H,h} & A_{S,h} \\ A^\top_{H,h} & A_{H,H} & A_{S,H} \\ A^\top_{S,h} & A^\top_{S,H} & A_{S,S} \end{pmatrix} \begin{pmatrix} \underline{u}_h \\ \underline{u}_H \\ \underline{u}_S \end{pmatrix} = \begin{pmatrix} \underline{f}_h \\ \underline{f}_H \\ \underline{f}_S \end{pmatrix} \tag{5.46}$$

Setting

$$A_{L,L} := \begin{pmatrix} A_{h,h} & A_{H,h} \\ A^\top_{H,h} & A_{H,H} \end{pmatrix}, \quad A_{S,L} := \begin{pmatrix} A_{S,h} \\ A_{S,H} \end{pmatrix}, \quad \underline{f}_L := \begin{pmatrix} \underline{f}_h \\ \underline{f}_H \end{pmatrix}.$$

the linear system (5.46) can be written as

$$\begin{pmatrix} A_{L,L} & A_{S,L} \\ A^\top_{S,L} & A_{S,S} \end{pmatrix} \begin{pmatrix} \underline{u}_L \\ \underline{u}_S \end{pmatrix} = \begin{pmatrix} \underline{f}_L \\ \underline{f}_S \end{pmatrix} \tag{5.47}$$

where the $\underline{u}_L = (\underline{u}_{L,i})_{i=1}^p$ denotes the local degrees of freedom which are separated within the subdomains Ω_i. The linear system (5.47) corresponds to the standard system arising in finite element domain decomposition methods [17, 41]. In particular, when using only the coarse grid space X_H, our approach coincides with the standard one. However, since we are using fine grid trial spaces locally, this gives an improvement of the proposed method. Note that this approach is a realization of the more general theory described in the previous section. Hence we can apply the stability and error analysis developed there. To see this, we eliminate in (5.47) the local degrees of freedom to get the Schur complement system

$$\Big[A_{S,S} - A_{S,L}^\top A_{L,L}^{-1} A_{S,L}\Big] \underline{u}_S = \underline{f}_S - A_{S,L}^\top A_{L,L}^{-1} \underline{f}_L . \tag{5.48}$$

Note that (5.48) is equivalent to the Schur complement system (5.35). Therefore, the linear system (5.48) represents a two–level finite element approximation of the variational problem (5.14). Hence we can apply the general theory given in the previous chapter to get stability of (5.48) as well as an error estimate as given in (5.29).

5.3 Three–Field Methods

Dirichlet domain decomposition methods as discussed in Section 5.1 are based on a strong coupling of the Dirichlet data, and a weak coupling of the Neumann data, see (5.13). However, by inserting the local Dirichlet–Neumann maps into (5.13), the Neumann data were eliminated. In many applications, however, one is interested to keep the local Neumann data as dual variables. Starting from (5.14) we will describe a domain decomposition method, which couples the local Cauchy data with the global Dirichlet data on the skeleton. This three–field algorithm, which corresponds to a two–fold saddle point problem as described in Section 1.2, was first introduced in [24] in the context of a finite element domain decomposition method.

Let us consider the variational problem (5.14) where for $v \in H^{1/2}(\Gamma_S)$ we used the pointwise restriction (5.12) to define the localized function $v_i := v_{|\Gamma_i} \in H^{1/2}(\Gamma_i)$. Instead of (5.12) we may use local trace operators

$$\gamma_0^i \,:\, H^{1/2}(\Gamma_S) \to H^{1/2}(\Gamma_i) \quad \text{for}\, i = 1, \ldots, p. \tag{5.49}$$

Then, the variational problem (5.14) reads: find $\widetilde{u} \in H^{1/2}(\Gamma_S)$, $\widetilde{u}(x) = g_D(x)$ for $x \in \Gamma_D$, such that

$$\sum_{i=1}^p \int_{\Gamma_i} \widetilde{S}_i \gamma_0^i \widetilde{u}_i(x) \gamma_0^i v(x) ds_x = \sum_{i=1}^p \int_{\Gamma_i} \widetilde{N}_i f(x) \gamma_0^i v(x) ds_x + \int_{\Gamma_N} g_N(x) v(x) ds_x$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma_D)$. For $\mu_i \in H^{-1/2}(\Gamma_i)$ we can also define the adjoint operator $\gamma_0^{i,*} : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_S)$ satisfying

$$\langle \gamma_0^{i,*} \mu_i, v \rangle_{L_2(\Gamma_S)} = \langle \mu_i, \gamma_0^i v \rangle_{L_2(\Gamma_i)} \quad \text{for all } v \in H^{1/2}(\Gamma_S). \tag{5.50}$$

Hence we can write the variational problem (5.14) as follows: find $\widetilde{u} \in H^{1/2}(\Gamma_S)$ with $\widetilde{u}(x) = g_D(x)$ for $x \in \Gamma_D$ such that

$$\sum_{i=1}^{p} \langle \gamma_0^{i,*} \widetilde{S}_i \gamma_0^i \widetilde{u}, v \rangle_{L_2(\Gamma_i)} = \sum_{i=1}^{p} \langle \gamma_0^{i,*} \widetilde{N}_i f, v \rangle_{L_2(\Gamma_i)} + \langle g_N, v \rangle_{L_2(\Gamma_N)} \tag{5.51}$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma_D)$. Since the variational problem (5.51) corresponds to (5.14) in its continuous form, we have unique solvability of (5.51). However, the bilinear form in (5.51) is a composition of three operators locally. In (5.20) we have defined local trial spaces $X_{h,i}$ by restriction of the global trial space, in particular $X_{h,i} = \gamma_0^i X_h$. Then it was only necessary to discretize the local Steklov–Poincaré operators $\widetilde{S}_i$. Here we will retain the structure of the locally composed operators. This allows us to introduce local trial spaces $X_{h,i}$ in a more general way.

For $\widetilde{u} \in H^{1/2}(\Gamma_S)$ we define

$$\widetilde{u}_i := \gamma_0^i \widetilde{u} \in H^{1/2}(\Gamma_i), \quad \widetilde{\lambda}_i := \widetilde{S}_i \widetilde{u}_i - \widetilde{N}_i f \in H^{-1/2}(\Gamma_i).$$

Considering the weak formulation of these relations, the variational problem (5.51) finally reads: find $\widetilde{u} \in H^{1/2}(\Gamma_S)$ with $\widetilde{u}(x) = g_D(x)$ for $x \in \Gamma_D$, $\widetilde{u}_i \in H^{1/2}(\Gamma_i)$ and $\widetilde{\lambda}_i \in H^{-1/2}(\Gamma_i)$ such that

$$\sum_{i=1}^{p} \langle \gamma_0^{i,*} \widetilde{\lambda}_i, v \rangle_{L_2(\Gamma_i)} = \langle g_N, v \rangle_{L_2(\Gamma_N)} \tag{5.52}$$

$$\langle \widetilde{S}_i \widetilde{u}_i, v_i \rangle_{L_2(\Gamma_i)} - \langle \widetilde{\lambda}_i, v_i \rangle_{L_2(\Gamma_i)} = \langle \widetilde{N}_i f, v_i \rangle_{L_2(\Gamma_i)} \tag{5.53}$$

$$\langle \widetilde{u}_i, \mu_i \rangle_{L_2(\Gamma_i)} - \langle \gamma_0^i \widetilde{u}, \mu_i \rangle_{L_2(\Gamma_i)} = 0 \tag{5.54}$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma_D)$, $v_i \in H^{1/2}(\Gamma_i)$ and $\mu_i \in H^{-1/2}(\Gamma_i)$. Let us define the function spaces

$$X := \prod_{i=1}^{p} H^{1/2}(\Gamma_i), \quad \Pi_1 := \prod_{i=1}^{p} H^{-1/2}(\Gamma_i), \quad \Pi_2 := H_0^{1/2}(\Gamma_S, \Gamma_D)$$

and the bounded bilinear forms

$$a(\underline{u}, \underline{v}) := \sum_{i=1}^{p} \langle \widetilde{S}_i u_i, v_i \rangle_{L_2(\Gamma_i)},$$

$$b_1(\underline{v}, \underline{\mu}) := \sum_{i=1}^{p} \langle \mu_i, v_i \rangle_{L_2(\Gamma_i)},$$

$$b_2(\underline{\mu}, v) := \sum_{i=1}^{p} \langle \mu_i, \gamma_0^i v \rangle_{L_2(\Gamma_i)}.$$

Then, the variational problem (5.52)–(5.54) corresponds to the two–fold saddle point problem as considered in (1.26). Hence we have to check the assumptions of Theorem 1.5 to ensure unique solvability of (5.52)–(5.54).

To describe $\ker B_2$ we write

$$0 \;=\; b_2(\underline{\mu}, v) \;=\; \sum_{i=1}^{p} \langle \mu_i, \gamma_0^i v\rangle_{L_2(\Gamma_i)} \;=\; \sum_{i<j} \langle \mu_i + \mu_j, v_{|\Gamma_{ij}}\rangle_{L_2(\Gamma_{ij})} \quad \text{for all } v \in \Pi_2.$$

Hence we have

$$\ker B_2 \;=\; \big\{ \underline{\mu} \in \Pi_1 \;:\; \mu_i + \mu_j = 0 \text{ on } \Gamma_{ij}, i<j \big\}\,. \tag{5.55}$$

Now we can characterize $\ker_{B_2} B_1$. For $\underline{\mu} \in \ker B_2$ we have

$$\begin{aligned} 0 \;=\; b_1(v, \underline{\mu}) &= \sum_{i=1}^{p} \langle \mu_i, v_i\rangle_{L_2(\Gamma_i)} \\ &= \sum_{i=1}^{p} \langle \mu_i, v_i\rangle_{L_2(\Gamma_D \cap \Gamma_i)} + \sum_{i<j} [\langle \mu_i, v_i\rangle_{L_2(\Gamma_{ij})} + \langle \mu_j, v_j\rangle_{L_2(\Gamma_{ij})}] \\ &= \sum_{i=1}^{p} \langle \mu_i, v_i\rangle_{L_2(\Gamma_D \cap \Gamma_i)} + \sum_{i<j} \langle \mu_i, v_i - v_j\rangle_{L_2(\Gamma_{ij})}\,. \end{aligned}$$

Therefore,

$$\ker_{B_2} B_1 := \{ \underline{v} \in X : v_i(x) = 0 \text{ for } x \in \Gamma_i \cap \Gamma_D, v_i(x) = v_j(x) \text{ for } x \in \Gamma_{ij} \}\,. \tag{5.56}$$

Note that

$$\ker_{B_2} B_1 \;=\; H_0^{1/2}(\Gamma_S, \Gamma_D) \tag{5.57}$$

and we can conclude the ellipticity of the bilinear form $a(\cdot,\cdot)$ on $\ker_{B_2} B_1$. Now, applying Theorem 1.5 we have unique solvability of the three–field variational problem (5.52)–(5.54).

For a Galerkin discretization of (5.52)–(5.54) we introduce finite–dimensional trial spaces,

$$X_h := \prod_{i=1}^{p} X_{h,i} \quad \text{with } X_{h,i} \;:=\; \text{span}\{\varphi_j^i\}_{j=1}^{M_i} \subset H^{1/2}(\Gamma_i),$$

$$\Pi_{1,h} := \prod_{i=1}^{p} \Pi_{1,h,i} \quad \text{with } \Pi_{1,h,i} \;:=\; \text{span}\{\chi_\ell^i\}_{\ell=1}^{N_i} \subset H^{-1/2}(\Gamma_i),$$

$$\Pi_{2,h} := \text{span}\{\varphi_k\}_{k=1}^{M} \subset H_0^{1/2}(\Gamma_S, \Gamma_D).$$

Let $\widetilde{g}_D \in H^{1/2}(\Gamma_S)$ be a bounded extension of the given Dirichlet datum g_D. The Galerkin variational problem of (5.52)–(5.54) then reads: find $\widetilde{u}_{0,h} \in \Pi_{2,h}$, $\widetilde{u}_{i,h} \in X_{h,i}$, $\widetilde{\lambda}_{i,h} \in \Pi_{1,h,i}$ such that

$$\sum_{i=1}^{p} \langle \gamma_0^{i,*} \widetilde{\lambda}_{i,h}, v_h \rangle_{L_2(\Gamma_i)} = \langle g_N, v_h \rangle_{L_2(\Gamma_N)} \tag{5.58}$$

$$\langle \widetilde{S}_i \widetilde{u}_{i,h}, v_{i,h} \rangle_{L_2(\Gamma_i)} - \langle \widetilde{\lambda}_{i,h}, v_{i,h} \rangle_{L_2(\Gamma_i)} = \langle \widetilde{N}_i f, v_{i,h} \rangle_{L_2(\Gamma_i)} \tag{5.59}$$

$$\langle \widetilde{u}_{i,h}, \mu_{i,h} \rangle_{L_2(\Gamma_i)} - \langle \gamma_0^i \widetilde{u}_{0,h}, \mu_{i,h} \rangle_{L_2(\Gamma_i)} = \langle \gamma_0^i \widetilde{g}_D, \mu_{i,h} \rangle_{L_2(\Gamma_i)} \tag{5.60}$$

for all $\widetilde{v}_h \in \Pi_{2,h}$, $v_{i,h} \in X_{h,i}$, $\mu_{i,h} \in \Pi_{1,h,i}$.

To ensure unique solvability and stability of (5.58)–(5.60) we apply Theorem 1.6. Hence we need to have the inf–sup conditions (1.34) and (1.35) to be satisfied as well as the ellipticity of the bilinear form $a(\cdot,\cdot)$ on $\ker_{B_{2,h}} B_{1,h}$ with

$$\ker B_{2,h} = \{\underline{\mu}_h \in \Pi_{1,h} \, : \, \sum_{i<j} \langle \mu_{i,h} + \mu_{j,h}, v_h \rangle_{L_2(\Gamma_{ij})} = 0 \text{ for all } v_h \in \Pi_{2,h}\},$$

$$\ker_{B_{2,h}} B_{1,h} =$$

$$= \{\underline{v}_h \in X_h : \sum_{i<j} \int_{\Gamma_{ij}} (\mu_{i,h} v_{i,h} + \mu_{j,h} v_{j,h}) ds_x = 0 \, \text{for all} \, \underline{\mu}_h \in \ker B_{2,h}\}.$$

Lemma 5.4. *Let us assume*

$$ker\, S_i \subset X_{h,i}, \quad ker\, S_{i|\Gamma_i \setminus \Gamma_D} \subset \Pi_{2,h|\Gamma_i \setminus \Gamma_D}. \tag{5.61}$$

Then, the bilinear form $a(\cdot,\cdot)$ is elliptic on $ker_{B_{2,h}} B_{1,h}$,

$$\begin{aligned} a(\underline{v}_h, \underline{v}_h) &= \sum_{i=1}^{p} \langle \widetilde{S}_i v_{i,h}, v_{i,h} \rangle_{L_2(\Gamma_i)} \\ &\geq c \cdot ||\underline{v}_h||^2_{H^{1/2}(\Gamma_S)} \quad \textit{for all}\, \underline{v}_h \in ker_{B_{2,h}} B_{1,h}. \end{aligned} \tag{5.62}$$

Proof. Let $v_{i,h} \in \ker S_i \cap \ker_{B_{2,h}} B_{1,h}$. Due to the assumptions made above we then have $v_{i,h} \in \Pi_{2,h}$ yielding $v_{i,h} = v_{j,h}$ on Γ_{ij}. Applying these arguments recursively we get $v_{j,h} \in \ker S_j$ for some j with $\Gamma_j \cap \Gamma_D \neq \emptyset$. Hence, $v_{h,j} \equiv 0$ and therefore $\underline{v}_h \equiv 0$. From this we can conclude (5.62). □

Let us now consider (1.34),

$$\frac{1}{c_S} \cdot ||\underline{\mu}_h||_{\Pi_1} \leq \sup_{0 \neq \underline{v}_h \in X_h} \frac{\sum_{i=1}^{p} \langle \mu_{h,i}, v_{h,i} \rangle_{L_2(\Gamma_i)}}{||\underline{v}_h||_X} \quad \text{for all } \underline{\mu}_h \in \Pi_{1,h}. \tag{5.63}$$

Lemma 5.5. *Let the discrete inf–sup condition*

$$\frac{1}{c_S} \cdot ||\mu_{h,i}||_{H^{-1/2}(\Gamma_i)} \leq \sup_{0 \neq v_{h,i} \in X_{h,i}} \frac{\langle \mu_{h,i}, v_{h,i} \rangle_{L_2(\Gamma_i)}}{||v_{h,i}||_{H^{1/2}(\Gamma_i)}} \quad \textit{for all}\, \mu_{h,i} \in \Pi_{1,h,i} \tag{5.64}$$

be satisfied for $i = 1, \ldots, p$. Then, (5.63) is valid.

Proof. For every $\mu_{h,i} \in \Pi_{1,h,i}$ we define $v^*_{h,i} := \widetilde{Q}_{h,i}\mu_{h,i}$,

$$\langle v^*_{h,i}, \eta_{h,i}\rangle_{L_2(\Gamma_i)} = \langle \widetilde{Q}_{h,i}\mu_{h,i}, \eta_{h,i}\rangle_{L_2(\Gamma_i)} = \langle \mu_{h,i}, \eta_{h,i}\rangle_{H^{-1/2}(\Gamma_i)}$$

for all $\eta_{h,i} \in \Pi_{1,h,i}$. Choosing $\eta_{h,i} = \mu_{h,i}$ this gives

$$||\mu_{h,i}||^2_{H^{-1/2}(\Gamma_i)} = \langle \mu_{h,i}, v^*_{h,i}\rangle_{L_2(\Gamma_i)}$$

as well as

$$||\underline{\mu}_h||^2_{\Pi_1} = \sum_{i=1}^{p} ||\mu_{h,i}||^2_{H^{-1/2}(\Gamma_i)} = b_1(\underline{v}_h, \underline{\mu}_h).$$

Using duality,

$$\begin{aligned} ||v^*_{h,i}||_{H^{1/2}(\Gamma_i)} &= \sup_{0\neq\eta_i\in H^{-1/2}(\Gamma_i)} \frac{|\langle v^*_{h,i}, \eta_i\rangle_{L_2(\Gamma_i)}|}{||\eta_i||_{H^{-1/2}(\Gamma_i)}} \\ &= \sup_{0\neq\eta_i\in H^{-1/2}(\Gamma_i)} \frac{|\langle v^*_{h,i}, \widetilde{Q}^*_h\eta_i\rangle_{L_2(\Gamma_i)}|}{||\eta_i||_{H^{-1/2}(\Gamma_i)}} \\ &= \sup_{0\neq\eta_i\in H^{-1/2}(\Gamma_i)} \frac{|\langle \mu_{h,i}, \widetilde{Q}^*_h\eta_i\rangle_{H^{-1/2}(\Gamma_i)}|}{||\eta_i||_{H^{-1/2}(\Gamma_i)}} \le c_S \cdot ||\mu_{h,i}||_{H^{-1/2}(\Gamma_i)}. \end{aligned}$$

Therefore,

$$||\underline{v}_h||^2_X = \sum_{i=1}^{p} ||v_{h,i}||^2_{H^{1/2}(\Gamma_i)} \le c_S^2 \cdot \sum_{i=1}^{p} ||\mu_{h,i}||^2_{H^{-1/2}(\Gamma_i)} = c_S^2 \cdot ||\underline{\mu}_h||^2_{\Pi_1}$$

and the assertion follows. □

It remains to justify (1.35),

$$\frac{1}{c_S} \cdot ||v_h||_{\Pi_2} \le \sup_{0\neq\underline{\mu}_h\in\Pi_{1,h}} \frac{\sum_{i=1}^{p} \langle \mu_{h,i}, \gamma_0^i v\rangle_{L_2(\Gamma_i)}}{||\underline{\mu}_h||_{\Pi_1}} \quad \text{for all } v_h \in \Pi_{2,h}\,. \tag{5.65}$$

Let $\Pi_{2,h,i} := \gamma_0^i \Pi_{2,h}$ the restriction of the global trial space $\Pi_{2,h}$ onto the local subdomain boundary Γ_i. Instead of (5.65) we now consider the local inf–sup conditions

$$\frac{1}{c_S} \cdot ||v_h^i||_{H^{1/2}(\Gamma_i)} \le \sup_{0\neq\mu_{h,i}\in\Pi_{1,h,i}} \frac{\langle \mu_{h,i}, v_h^i\rangle_{L_2(\Gamma_i)}}{||\mu_{h,i}||_{H^{-1/2}(\Gamma_i)}} \quad \text{for all } v_h^i \in \Pi_{2,h,i}. \tag{5.66}$$

As in Lemma 5.5 the local inf–sup conditions (5.66) imply the global stability condition (5.65). Hence we have to define, for a global coarse grid space $\Pi_{2,h}$, the local trial spaces $X_{h,i}$ and $\Pi_{1,h,i}$ in such a way, that the discrete inf–sup conditions (5.64) and (5.66) are satisfied, see Chapter 2.

Let us define the Galerkin matrices

$$\begin{aligned}
\widetilde{S}_{h,i}[\widetilde{j},j] &= \langle \widetilde{S}_i \varphi^i_j, \varphi^i_{\widetilde{j}} \rangle_{L_2(\Gamma_i)} && j,\widetilde{j} = 1,\ldots,M_i \\
M_{h,i}[\ell,j] &= \langle \varphi^i_j, \chi^i_\ell \rangle_{L_2(\Gamma_i)} && j = 1,\ldots,M_i; \ell = 1,\ldots,N_i \\
\bar{M}_{h,i}[\ell,k] &= \langle \gamma^i_0 \varphi_k, \chi^i_\ell \rangle_{L_2(\Gamma_i)} && k = 1,\ldots,M; \ell = 1,\ldots,N_i
\end{aligned}$$

locally as well as local vectors

$$\begin{aligned}
g_{N,k} &= \langle g_N, \varphi_k \rangle_{L_2(\Gamma_N)} && k = 1,\ldots,M; \\
f^i_j &= \langle \widetilde{N}_i f, \varphi^i_j \rangle_{L_2(\Gamma_i)} && j = 1,\ldots,M_i; \\
g^i_{D,\ell} &= \langle \gamma^i_0 \widetilde{g}_D, \chi^i_\ell \rangle_{L_2(\Gamma_i)} && \ell = 1,\ldots,N_i.
\end{aligned}$$

Now, (5.58)–(5.60) is equivalent to the system of linear equations,

$$\begin{pmatrix} 0 & M_h & -\bar{M}_h \\ -M_h^\top & \widetilde{S}_h & 0 \\ \bar{M}_h^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\widetilde{\lambda}} \\ \underline{\widetilde{u}} \\ \underline{\widetilde{u}}_0 \end{pmatrix} = \begin{pmatrix} \underline{g}_D \\ \underline{f} \\ \underline{g}_N \end{pmatrix} \tag{5.67}$$

where for the discrete Steklov–Poincaré operator $\widetilde{S}_h$ we may insert either the finite element representation (4.17) or the boundary element representation (4.20).

For an efficient iterative solution of (5.67) we can adapt the approach described in [16] to transform the block–skew symmetric and positive definite stiffness matrix in (5.67) to a symmetric and positive definite matrix.

When assuming the stability condition (5.63) and

$$M_i = \dim X_{h,i} = \dim \Pi_{1,h,i} = N_i \quad \text{for } i = 1,\ldots,p,$$

the local matrices $M_{h,i}$ are invertible and we can first eliminate $\underline{\widetilde{u}}$ in (5.67),

$$\underline{\widetilde{u}} = M_h^{-1} \left[\bar{M}_h \underline{\widetilde{u}}_0 + \underline{g}_D \right]$$

to get

$$M_h^\top \underline{\widetilde{\lambda}} = \widetilde{S}_h M_h^{-1} \left[\bar{M}_h \underline{\widetilde{u}}_0 + \underline{g}_D \right] - \underline{f}$$

or

$$\underline{\widetilde{\lambda}} = M_h^{-\top} \widetilde{S}_h M_h^{-1} \bar{M}_h \underline{\widetilde{u}}_0 + M_h^{-\top} \widetilde{S}_h M_h^{-1} \underline{g}_D - M_h^{-\top} \underline{f}.$$

Therefore we have to solve the Schur complement system

$$\bar{M}_h^\top M_h^{-\top} \widetilde{S}_h M_h^{-1} \bar{M}_h \underline{\widetilde{u}}_0 = \underline{g}_N - \bar{M}^\top M_h^{-\top} \widetilde{S}_h M_h^{-1} \underline{g}_D + \bar{M}_h^\top M_h^{-\top} \underline{f} \tag{5.68}$$

which is the three–field approximation of (5.51).

5.4 Neumann Domain Decomposition Methods

In this section we consider domain decomposition methods which are based on a weak coupling of the Dirichlet data and a strong coupling of the associated Neumann data, see also [44]. For neighbored subdomains Ω_i and Ω_j with a local coupling boundary $\Gamma_{ij}, i < j$ we introduce functions $\widetilde{\lambda}_{ij} \in H^{-1/2}(\Gamma_{ij})$. Then we define, for $x \in \Gamma_{ij}$,

$$\widetilde{\lambda}_i(x) := \widetilde{\lambda}_{ij}(x) \quad \text{for } i < j, \qquad \widetilde{\lambda}_j(x) := -\widetilde{\lambda}_{ij}(x) \quad \text{for } j > i \tag{5.69}$$

and

$$\widetilde{\lambda}_i(x) := g_N(x) \quad \text{for } x \in \Gamma_i \cap \Gamma_N. \tag{5.70}$$

Note that $\widetilde{\lambda}_i \in H^{-1/2}(\Gamma_i)$ when assuming $\widetilde{\lambda}_{ij} \in H^{-1/2}(\Gamma_{ij})$. Obviously, the Neumann boundary and transmissions conditions in (5.11) are satisfied for this choice. Then the weak coupling conditions for the Dirichlet data are

$$\sum_{i<j}^{p} \int_{\Gamma_{ij}} [\widetilde{u}_i(x) - \widetilde{u}_j(x)]\mu_{ij}(x) ds_x = 0 \quad \text{for all } \mu_{ij} \in H^{-1/2}(\Gamma_{ij}). \tag{5.71}$$

Since we are using $\mu_{ij} \in H^{-1/2}(\Gamma_{ij})$, we have to require

$$\widetilde{u}_i - \widetilde{u}_j \in \widetilde{H}^{1/2}(\Gamma_{ij}). \tag{5.72}$$

Note that this is a quite restrictive assumption, especially for $n = 3$. For $n = 2$, (5.72) is equivalent to continuity at the cross points, while for $n = 3$ we need also continuity across the edges.

Let us define the function spaces

$$X := \left\{ \underline{v} \in \prod_{i=1}^{p} H^{1/2}(\Gamma_i) \,:\, v_i - v_j \in \widetilde{H}^{1/2}(\Gamma_{ij}), v_i(x) = 0 \text{ for } x \in \Gamma_D \right\} \tag{5.73}$$

$$\Pi := \prod_{i=1}^{p} H^{-1/2}(\Gamma_i \cap \Gamma_N) \times \prod_{i<j} H^{-1/2}(\Gamma_{ij}) \tag{5.74}$$

which are equipped with the norms

$$||\underline{v}||_X := \left\{ \sum_{i=1}^{p} ||v_i||^2_{H^{1/2}(\Gamma_i)} \right\}^{1/2}, \tag{5.75}$$

$$||\underline{\mu}||_\Pi := \left\{ \sum_{i=1}^{p} ||\mu_i||^2_{H^{-1/2}(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} ||\mu_{ij}||^2_{H^{-1/2}(\Gamma_{ij})} \right\}^{1/2}. \tag{5.76}$$

Moreover, we consider the bounded bilinear forms

$$a(\underline{u},\underline{v}) := \sum_{i=1}^{p} \langle \widetilde{S}_i u_i, v_i \rangle_{L_2(\Gamma_i)} \; : \; X \times X \to \mathbb{R}, \tag{5.77}$$

$$b(\underline{v},\underline{\mu}) := \sum_{i=1}^{p} \langle u_i, \mu_i \rangle_{L_2(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} \langle v_i - v_j, \mu_{ij} \rangle_{L_2(\Gamma_{ij})} \; : \; X \times \Pi \to \mathbb{R}. \tag{5.78}$$

Let $\widetilde{g}_D \in H^{1/2}(\Gamma_S)$ be an arbitrary but fixed extension of the given Dirichlet data. Considering a weak formulation of the local Dirichlet–Neumann maps in (5.11) together with the weak coupling conditions (5.71) leads to the variational formulation: find $\underline{\widetilde{u}}_0 \in X$ and $\underline{\widetilde{\lambda}} \in \Pi$ such that

$$\begin{aligned} a(\underline{\widetilde{u}}_0, \underline{v}) - b(\underline{v}, \underline{\widetilde{\lambda}}) &= \sum_{i=1}^{p} \langle \widetilde{N}_i f - \widetilde{S}_i \widetilde{g}_D, v_i \rangle_{L_2(\Gamma_i)} \\ b(\underline{\widetilde{u}}_0, \underline{\mu}) &= 0 \end{aligned} \tag{5.79}$$

for all $\underline{v} \in X$ and $\underline{\mu} \in \Pi$. Note that

$$V \;:=\; \ker B \;=\; \{\underline{v} \in X \;:\; v_i(x) = v_j(x) \text{ for } x \in \Gamma_{ij}\} \;=\; H_0^{1/2}(\Gamma_S, \Gamma_D). \tag{5.80}$$

Hence we have

$$a(\underline{v},\underline{v}) \;=\; \sum_{i=1}^{p} \langle \widetilde{S}_i v_i, v_i \rangle_{L_2(\Gamma_i)} \;\geq\; c_1^S \cdot ||\underline{v}||_X^2 \quad \text{for all } v \in V. \tag{5.81}$$

It remains to check the inf–sup condition (1.18) to get unique solvability of (5.79).

Theorem 5.6. *The bilinear form $b(\cdot,\cdot) : X \times \Pi$ defined by (5.78) satisfies the inf–sup condition (1.18).*

Proof. For an arbitrary but fixed $\underline{\mu} \in \Pi$ we define $\underline{u}^* \in X$ as follows: For $i < j$ let $u_{ij}^* \in \widetilde{H}^{1/2}(\Gamma_{ij})$ be the unique solution solution of the variational problem

$$\langle u_{ij}^*, v_{ij} \rangle_{\widetilde{H}^{1/2}(\Gamma_{ij})} \;=\; \langle \mu_{ij}, v_{ij} \rangle_{L_2(\Gamma_{ij})} \quad \text{for all } v_{ij} \in \widetilde{H}^{1/2}(\Gamma_{ij})$$

and let $u_{ji}^* = 0$ on Γ_{ij}. Note that

$$||u_{ij}^*||_{\widetilde{H}^{1/2}(\Gamma_{ij})} \;=\; ||\mu_{ij}||_{H^{-1/2}(\Gamma_{ij})} \quad \text{for } i < j.$$

For $\Gamma_i \cap \Gamma_N \neq \emptyset$ we define $u_{i,N}^* \in \widetilde{H}^{1/2}(\Gamma_i \cap \Gamma_N)$ such that

$$\langle u_{i,N}^*, v_{i,N} \rangle_{\widetilde{H}^{1/2}(\Gamma_i \cap \Gamma_N)} \;=\; \langle \mu_i, v_{ij} \rangle_{L_2(\Gamma_{ij})} \quad \text{for all } v_{ij} \in \widetilde{H}^{1/2}(\Gamma_{ij})$$

Again,

$$||u^*_{i,N}||_{\widetilde{H}^{1/2}(\Gamma_i \cap \Gamma_N)} = ||\mu_i||_{H^{-1/2}(\Gamma_i)}.$$

Then we define

$$u^*_i(x) = \begin{cases} u^*_{ij}(x) & \text{for } x \in \Gamma_{ij}, \\ u^*_{i,N}(x) & \text{for } x \in \Gamma_i \cap \Gamma_N, \\ 0 & \text{for } x \in \Gamma_i \cap \Gamma_D. \end{cases}$$

Note that

$$\begin{aligned} ||u^*_i||_{H^{1/2}(\Gamma_i)} &= ||u^*_{i,N} + \sum_j u^*_{i,j}||_{H^{1/2}(\Gamma_i)} \\ &\leq ||u^*_{i,N}||_{H^{1/2}(\Gamma_i \cap \Gamma_N)} + \sum_j ||u^*_{ij}||_{H^{/2}(\Gamma_{ij})} \end{aligned}$$

and therefore

$$||u^*_i||^2_{H^{1/2}(\Gamma_i)} \leq c_i \cdot \left\{ ||u^*_{i,N}||^2_{H^{1/2}(\Gamma_i \cap \Gamma_N)} + \sum_j ||u^*_{ij}||^2_{H^{/2}(\Gamma_{ij})} \right\}$$

where the constant c_i depends on the number of coupling boundaries locally. Then,

$$\begin{aligned} b(\underline{u}^*, \underline{\mu}) &= \sum_{i=1}^{p} \langle u^*_{i,N}, \mu_i \rangle_{L_2(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} \int_{\Gamma_{ij}} [u^*_i(x) - u^*_j(x)] \mu_{ij}(x) ds_x \\ &= \sum_{i=1}^{p} \langle u^*_{i,N}, \mu_i \rangle_{L_2(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} \int_{\Gamma_{ij}} u^*_{ij}(x) \mu_{ij}(x) ds_x \\ &= \sum_{i=1}^{p} ||u^*_{i,N}||^2_{\widetilde{H}^{1/2}(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} ||u^*_{ij}||^2_{\widetilde{H}^{1/2}(\Gamma_{ij})} \\ &= \left\{ \sum_{i=1}^{p} ||u^*_{i,N}||^2_{\widetilde{H}^{1/2}(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} ||u^*_{ij}||^2_{\widetilde{H}^{1/2}(\Gamma_{ij})} \right\}^{1/2} \\ &\quad \cdot \left\{ \sum_{i=1}^{p} ||\mu_i||^2_{H^{-1/2}(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} ||\mu_{ij}||^2_{H^{-1/2}(\Gamma_{ij})} \right\}^{1/2} \\ &\geq c \cdot ||\underline{u}^*||_{H^{1/2}(\Gamma_S)} ||\underline{\mu}||_{\Pi} \end{aligned}$$

implying the inf–sup condition (1.18). □

Hence we have unique solvability of the saddle point problem (5.79) due to Theorem 1.2.

For a Galerkin discretization of (5.79) we introduce trial spaces

$$X_h := \prod_{i=1}^{p} X_{h,i}, \quad \Pi_h := \prod_{i=1}^{p} \Pi_{i,N} \times \prod_{i<j} \Pi_{h,ij}$$

with

$$X_{h,i} := \operatorname{span}\{\varphi_k^i\}_{k=1}^{M_i} \subset H_0^{1/2}(\Gamma_i, \Gamma_D),$$

$$\Pi_{i,N} := \operatorname{span}\{\chi_\ell^{i,N}\}_{\ell=1}^{N_{i,N}} \subset H^{-1/2}(\Gamma_i \cap \Gamma_N),$$

$$\Pi_{h,ij} := \operatorname{span}\{\chi_\ell^{ij}\}_{\ell=1}^{N_{ij}} \subset H^{-1/2}(\Gamma_{ij}).$$

Note that the definition of $X_h \subset X$ includes the compatibility condition

$$u_{h,i} - u_{h,j} \in \widetilde{H}^{1/2}(\Gamma_{ij})\,. \tag{5.82}$$

The Galerkin problem of (5.79) is: find $\underline{\widetilde{u}}_{0,h} \in X_h$ and $\underline{\widetilde{\lambda}}_h \in \Pi_h$ such that

$$\begin{aligned} a(\underline{\widetilde{u}}_{0,h}, \underline{v}_h) - b(\underline{v}, \underline{\widetilde{\lambda}}_h) &= \sum_{i=1}^{p} \langle \widetilde{N}_i f - \widetilde{S}_i \widetilde{g}_D, v_{i,h} \rangle_{L_2(\Gamma_i)} \\ b(\underline{\widetilde{u}}_{0,h}, \underline{\mu}_h) &= 0 \end{aligned} \tag{5.83}$$

for all $\underline{v}_h \in X_h$ and $\underline{\mu}_h \in \Pi_h$.

To ensure unique solvability and stability of (5.83) we will apply Theorem 1.3. Hence we have to establish the discrete inf–sup condition (1.25),

$$\widetilde{\gamma}_S \cdot ||\underline{\mu}_h||_\Pi \le \sup_{0 \ne \underline{v}_h \in X_h} \frac{1}{||\underline{v}_h||_X} \left[\sum_{i=1}^{p} \langle v_i, \mu_i \rangle_{L_2(\Gamma_i \cap \Gamma_N)} + \sum_{i<j} \langle v_i - v_j, \mu_i \rangle_{L_2(\Gamma_{ij})} \right] \tag{5.84}$$

for all $\underline{\mu}_h \in \Pi_h$.

As in Lemma 5.5 we can prove the following result, see, for example, also [14].

Lemma 5.7. *Let the discrete inf–sup conditions*

$$\frac{1}{c_S} \cdot ||\mu_{h,ij}||_{H^{-1/2}(\Gamma_i \cap \Gamma_N)} \le \sup_{0 \ne v_{h,i} \in X_{h,i} \cap \widetilde{H}^{1/2}(\Gamma_{ij})} \frac{\langle \mu_{h,ij}, v_{h,i} \rangle_{L_2(\Gamma_{ij})}}{||v_{h,i}||_{\widetilde{H}^{1/2}(\Gamma_{ij})}} \tag{5.85}$$

for all $\mu_{h,ij} \in \Pi_{h,ij}$ and

$$\frac{1}{c_S} \cdot ||\mu_{h,i}||_{H^{-1/2}(\Gamma_{ij})} \le \sup_{0 \ne v_{h,i} \in X_{h,i} \cap \widetilde{H}^{1/2}(\Gamma_i \cap \Gamma_N)} \frac{\langle \mu_{h,i}, v_{h,i} \rangle_{L_2(\Gamma_i \cap \Gamma_N)}}{||v_{h,i}||_{\widetilde{H}^{1/2}(\Gamma_i \cap \Gamma_N)}} \tag{5.86}$$

for all $\mu_h, i \in \Pi_{i,N}$ be satisfied. Then, (5.84) is valid.

Proof. For simplicity in the presentation we consider the case $\Gamma_N = \emptyset$ only. If $\Gamma_N \neq \emptyset$, the additional terms correspond to the case considered for the coupling boundaries Γ_{ij}.

For $\mu_{h,ij} \in \Pi_{h,ij}$ and $i < j$ we define $v^*_{h,j}(x) = 0$ for $x \in \Gamma_{ij}$ and $v^*_{h,i} \in X_{h,i} \cap \widetilde{H}^{1/2}(\Gamma_{ij})$ satisfying

$$\langle v^*_{h,i}, \eta_{h,ij}\rangle_{L_2(\Gamma_{ij})} = \langle \widetilde{Q}_{h,ij}\mu_{h,ij}, \eta_{h,ij}\rangle_{L_2(\Gamma_{ij})} = \langle \mu_{h,ij}, \eta_{h,ij}\rangle_{H^{-1/2}(\Gamma_{ij})}$$

for all $\eta_{h,ij} \in \Pi_{h,ij}$. Then,

$$||\mu_{h,ij}||^2_{H^{-1/2}(\Gamma_{ij})} = \langle\langle v^*_{h,i}, \mu_{h,ij}\rangle_{L_2(\Gamma_{ij})}$$

as well as

$$||\underline{\mu}_h||^2_{\Pi} = \sum_{i<j} ||\mu_{h,ij}||^2_{H^{-1/2}(\Gamma_{ij})} = \sum_{i<j} \langle v^*_{h,i}, \mu_{h,ij}\rangle_{L_2(\Gamma_{ij})} = b(\underline{v}^*, \underline{\mu}_h).$$

Using duality and the stability of $\widetilde{Q}_{h,ij}$ we obtain

$$||v^*_{h,ij}||_{\widetilde{H}^{1/2}(\Gamma_{ij})} \leq c_S \cdot ||\mu_{h,ij}||_{H^{-1/2}(\Gamma_{ij})}$$

and therefore

$$||\underline{v}^*_h||^2_X = \sum_{i<j} ||v^*_{h,ij}||^2_{\widetilde{H}^{1/2}(\Gamma_{ij})} \leq c_S^2 \cdot \sum_{i<j} ||\mu_{h,ij}||^2_{H^{-1/2}(\Gamma_{ij})} = c_S^2 \cdot ||\underline{\mu}||^2_{\Pi}.$$

Now, the assertion follows. □

It remains to guarantee the discrete inf–sup condition (5.85). For $i < j$ we have to consider the trial space $X_{h,ij} := X_{h,i} \cap \widetilde{H}^{1/2}(\Gamma_{ij})$. To define $\Pi_{h,ij}$ we may first chose $\Pi_{h,ij} = X_{h,ij}$. Then, $\widetilde{Q}_{h,ij}$ corresponds to the L_2 projection operator Q_h as defined in (1.64). The stability of $\widetilde{Q}_{h,ij} : \widetilde{H}^{1/2}(\Gamma_{ij}) \to X_{h,ij} \subset \widetilde{H}^{1/2}(\Gamma_{ij})$ then follows from the local Assumption 2.1. In particular, when using piecewise linear basis functions to define the local trial spaces $X_{h,i}$, the mesh criteria (2.20) has to be satisfied, see the discussion and the examples in Section 2.1. Note that the stability results of $\widetilde{Q}_{h,ij}$ are valid for quite arbitrary meshes, i.e. we allow also adaptive and nonuniform meshes on the coupling boundary Γ_{ij} when the mesh assumption 2.1 is satisfied locally. The standard choice in Mortar finite element methods is $\Pi_{h,ij} = \widetilde{X}_{h,ij}$ where $\widetilde{X}_{h,ij}$ is modified as shown is Figure 5.3 for $n = 2$. To ensure the stability of $\widetilde{Q}_{h,ij}$ in this case we still have to consider the local trial spaces $V_h(\tau_1)$ and $V_h(\tau_N)$ only. Note that the local trial spaces $V_h(\tau_\ell)$ for $\ell = 2, \ldots, N-1$ correspond to the case already discussed above. Since the dimension of the local trial spaces $V_h(\tau_1)$ and $V_h(\tau_N)$ are equal to one, Assumptions 1.2 and Assumption 2.1 are trivially satisfied. Hence we obtain stability of $\widetilde{Q}_{h,ij}$ in $\widetilde{H}^{1/2}(\Gamma_{ij})$ when (2.20) is satisfied. For $n = 3$ we can define a corresponding modification of the piecewise linear basis functions on elements at the boundary of Γ_{ij}.

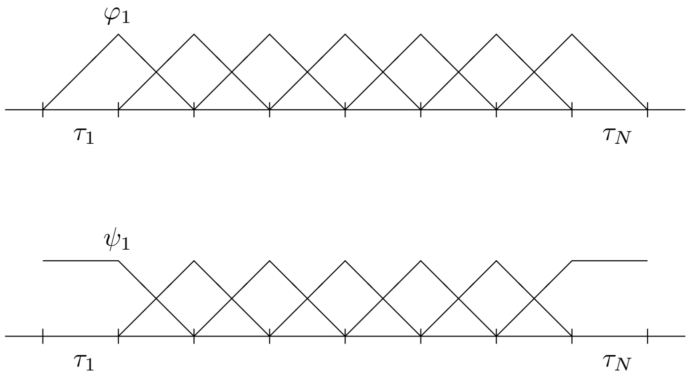


Figure 5.3: Local trial spaces $X_{h,ij}$ and $\Pi_{h,ij}$.

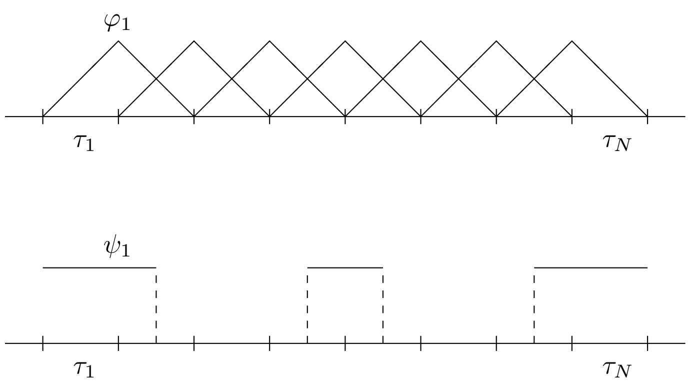


Figure 5.4: Local trial spaces $X_{h,ij}$ and $\Pi_{h,ij}$, dual basis.

Instead of piecewise linear basis functions to define $\Pi_{h,ij}$ we can also use piecewise constant basis functions which are defined on the dual mesh, see Figure 5.4 for $n = 2$. To get stability of $\widetilde{Q}_{h,ij}$ we can apply the results from Section 2.2, in particular (2.30). For $n = 3$, see Figure 2.3. Note, that this approach was also discussed in [49] assuming locally quasi-uniform meshes on Γ_{ij}.

Note that for the definition of the local trial spaces $X_{h,i}$ and $\Pi_{h,ij}$ we can consider also higher polynomial basis functions as discussed in Section 2.3 or biorthogonal basis functions, see [73] and Section 2.4. When using nonuniform meshes, then one has to control the local mesh assumption 2.1 by computing the eigenvalues of $\widetilde{G}_h^S$ (see (2.7)) numerically.

5.5 Numerical Results

In this Section we first present some numerical results to illustrate the applicability of the natural finite element domain decomposition method as introduced in Section 5.1.

As a model problem we consider the Dirichlet boundary value problem

$$-\mathrm{div}\,\alpha(x)\nabla u(x) \;=\; 1 \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \partial\Omega \tag{5.87}$$

where the domain Ω is given either by $\Omega_1 = (0,3)^2$ or by $\Omega_2 = (0,2)^2$. In case of Ω_1 the coefficient $\alpha(x)$ is given by (see Figure 5.5)

$$\alpha_1(x) \;=\; \begin{cases} 10^{-6} & \text{for } x \in (0.5, 2.5)^2 \backslash [1,2]^2, \\ 1 & \text{elsewhere,} \end{cases} \tag{5.88}$$

while in the case of Ω_2 we have (see Figure 5.6)

$$\alpha_2(x) \;=\; \begin{cases} 10^{-6} & \text{for } x \in (0,1)^2 \cup (1,2)^2, \\ 1 & \text{elsewhere.} \end{cases} \tag{5.89}$$

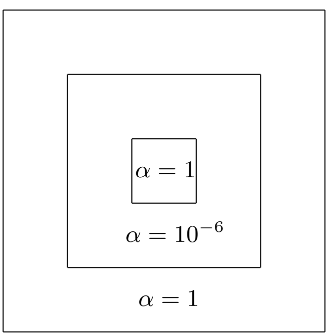


Figure 5.5: Domain Ω_1.

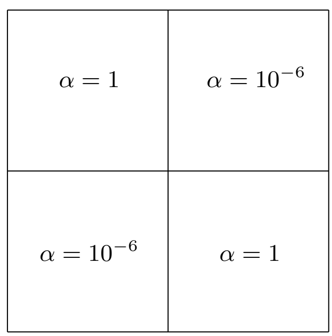


Figure 5.6: Domain Ω_2.

Both examples are standard test examples for Mortar finite element methods, see [73, 74]. The fine grid in the domains with coefficient $\alpha(x) = 10^{-6}$ is constructed by L_+ additional refinements starting from a global coarse grid with L initial refinement steps, see Figure 5.7 and Figure 5.8 with $L = 1$ and $L_+ = 3$.

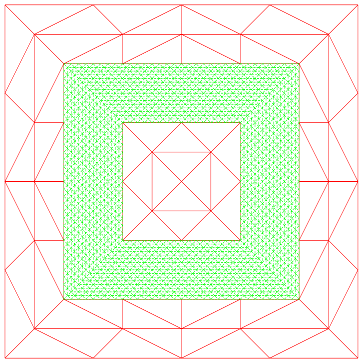


Figure 5.7: Triangulation of Ω_1.

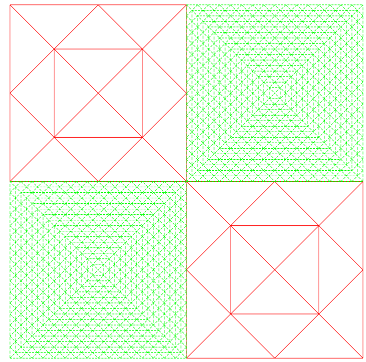


Figure 5.8: Triangulation of Ω_2.

The global trial space X_h and all local trial spaces $\widetilde{X}_{h,i}$ are defined by using piecewise linear basis functions. Since the coarse mesh is embedded in the fine mesh along the interfaces, the data transfer between the coarse and the fine mesh is realized by interpolation. In Figure 5.9 and Figure 5.10 we plot the solutions obtained along the diagonals from the left lower corner to the right upper one.

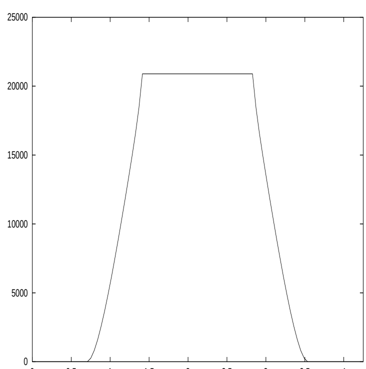


Figure 5.9:
Solution along the diagonal of Ω_1.

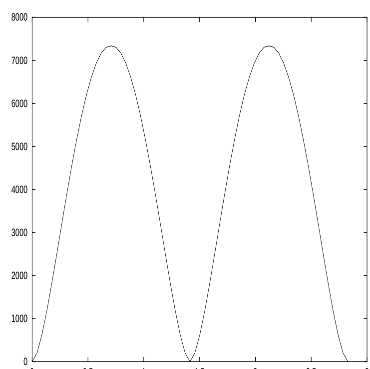


Figure 5.10:
Solution along the diagonal of Ω_2.

L	$L_+ = 0$		$L_+ = 1$		$L_+ = 2$		$L_+ = 3$		$L_+ = 4$	
0	13	2.73 -1	25	1.84 -1	85	1.09 -1	349	6.22 -2	1453	3.60 -2
1	57	1.84 -1	117	1.09 -1	381	6.22 -2	1485	3.60 -2	5997	2.12 -2
2	241	1.09 -1	505	6.22 -2	1609	3.59 -2	6121	2.12 -2	24361	1.27 -2
3	993	6.22 -2	2097	3.59 -2	6609	2.12 -2	24849	1.27 -2	98193	7.75 -3
4	4033	3.59 -2	8545	2.12 -2	26785	1.27 -2	100129	7.75 -3		
5	16257	2.12 -2	34497	1.27 -2	107841	7.75 -3				
6	65281	1.27 -2	138625	7.74 -3						
7	261633	(7.74 -3)								

Table 5.1. Degrees of freedom and approximate error for Ω_1.

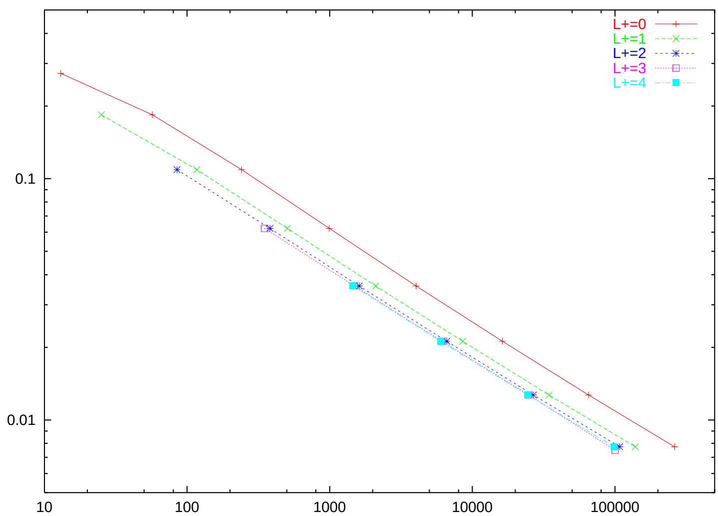


Figure 5.11: Error in energy for Ω_1.

L	$L_+ = 0$		$L_+ = 1$		$L_+ = 2$		$L_+ = 3$		$L_+ = 4$	
0	5	2.79 -1	13	3.22 -1	53	1.87 -1	229	9.86 -2	965	5.03 -2
1	25	3.22 -1	65	1.87 -1	241	9.85 -2	977	5.01 -2	3985	2.53 -2
2	113	1.87 -1	289	9.85 -2	1025	5.01 -2	4033	2.52 -2	16193	1.27 -2
3	481	9.85 -2	1217	5.01 -2	4225	2.52 -2	16385	1.26 -2	65281	6.34 -3
4	1985	5.01 -2	4993	2.52 -2	17153	1.26 -2	66049	6.31 -3		
5	8065	2.52 -2	20225	1.26 -2	69121	6.31 -3				
6	32513	1.26 -2	81409	6.31 -3						
7	130561	6.31 -3								

Table 5.2. Degrees of freedom and approximate error for Ω_2.

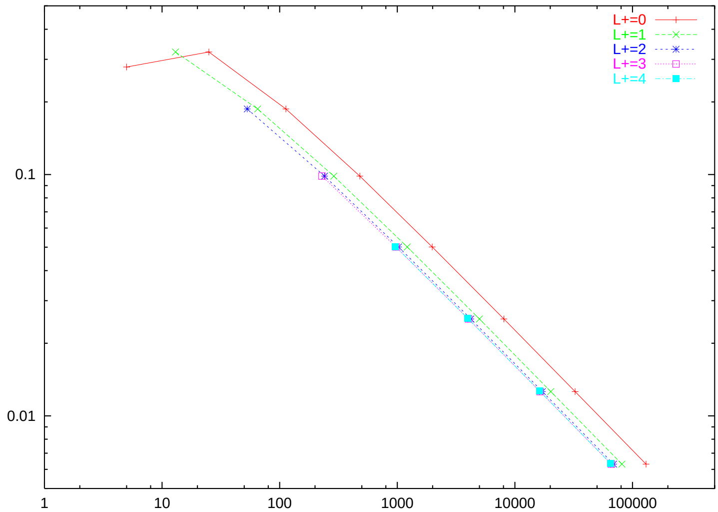


Figure 5.12: Error in energy norm for Ω_2.

Since the exact solution is not known for both test problems, we estimated an approximate error

$$e_h \;:=\; |u_h - u_{h/2}|_E$$

by computing an approximate solution on a refined mesh with respect to the energy norm

$$|v|_E^2 \;=\; \int\limits_{\Omega} \alpha(x)|\nabla v(x)|^2 dx \quad \text{in } H_0^1(\Omega, \Gamma).$$

The results are given for different values of L and L_+ in Table 5.1 and Table 5.2, see also Figure 5.11 and 5.12, respectively. Note that the number of degrees of freedom is reduced drastically by maintaining the same accuracy when using the approach described here.

As a second example we consider a model problem from linear elastostatics in three space dimensions. The equilibrium equations are

$$-\sigma_{ij,j}(u,x) \;=\; 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^3. \tag{5.90}$$

In the homogeneous, isotropic case the stress–strain relation is given by Hooke's law,

$$\sigma_{ij}(u,x) = \frac{E\nu}{(1+\nu)(1-2\nu)}\,\delta_{ij}\,\mathrm{div}u(x) + \frac{E}{1+\nu} e_{ij}(u,x) \tag{5.91}$$

with the linearized strain tensor

$$e_{ij}(u,x) = \frac{1}{2}\left[\frac{\partial}{\partial x_i} u_j(x) + \frac{\partial}{\partial x_j} u_i(x)\right]. \tag{5.92}$$

In (5.91), E is Young's modulus and ν is Poisson's ratio; here we used $E = 200$ and $\nu = 0.3$.

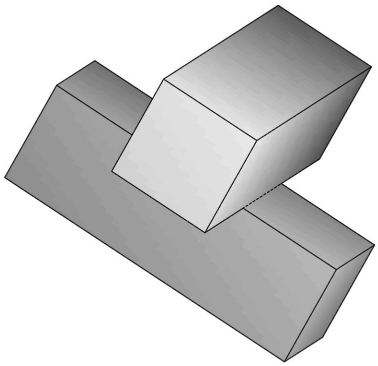


Figure 5.13:
Domain $\Omega \subset \mathbb{R}^3$.

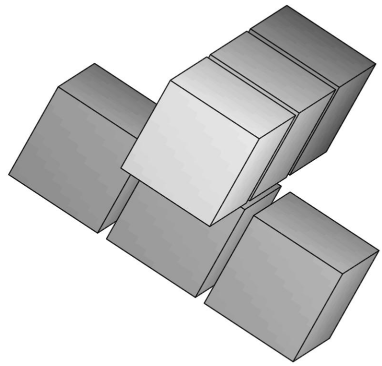


Figure 5.14
Domain decomposition into 6 substructures.

The domain Ω is given as depicted in Figure 5.13. Note that Ω fails to be a Lipschitz domain, see also [52]. But a simple domain decomposition as shown in Figure 5.14 leads to substructures which are Lipschitz. Note that a decomposition into two subdomains would be sufficient to obtain Lipschitz substructures.

The boundary conditions are as depicted in Figure 5.15: The lower bar is fixed at both ends while on the top of the upper bar a predescribed boundary stress density is given.

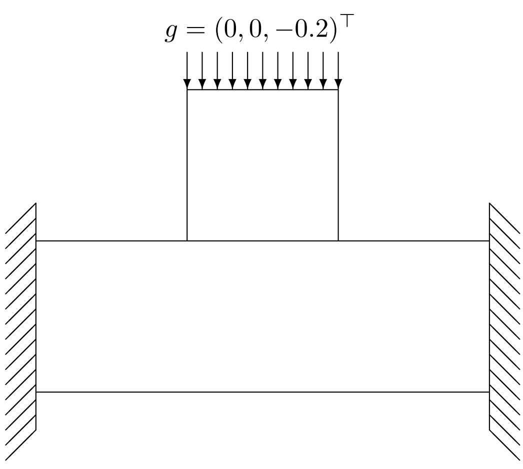


Figure 5.15: Boundary conditions.

This mixed boundary value problem is discretized by a domain decomposition approach using boundary elements as described in Section 5.1. Each substructure is discretized by 1536 boundary elements using piecewise constant basis functions to approximate the local Steklov–Poincaré operators (see Section 3.4, in particular (3.78)). To define the global trial space on the skeleton of the domain decomposition (see (5.19)) piecewise linear basis functions are used. This leads to a system of linear equations as given in (5.27) where the assembled stiffness matrix is symmetric and positive definite and the cg scheme can be used as solver. The deformed body is shown in Figure 5.16.

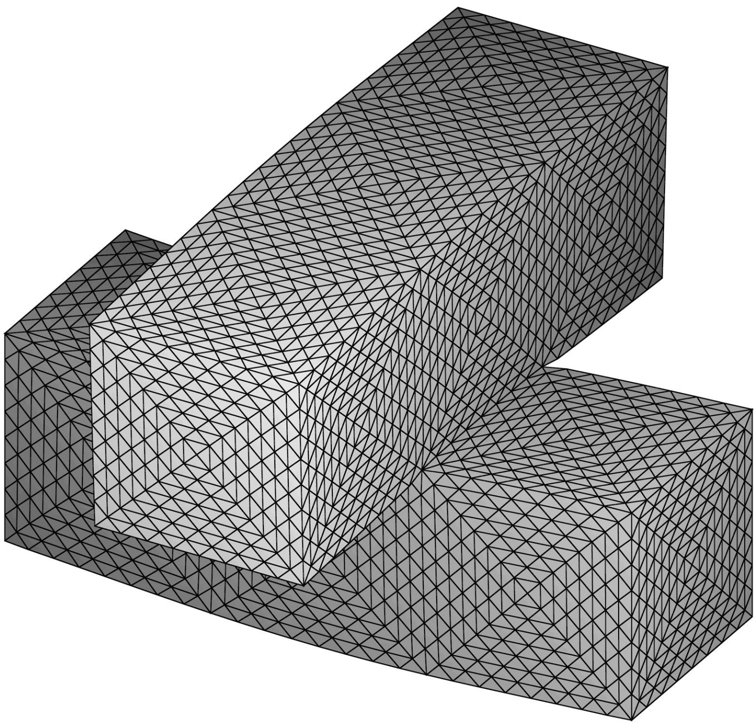


Figure 5.16: Deformed body with 6656 boundary elements.

5.6 Concluding Remarks

The main aim of this work was to construct stable domain decomposition methods which are based on different discretizations locally. We did not focus on efficient solution methods to solve the resulting linear algebraic systems in parallel, and we did not consider a posteriori error estimators to drive adaptive algorithms. However, our algorithms are especially designed to deal with adaptive and nonuniform meshes, even on local coupling boundaries. For an adaptive algorithm, we have to combine this approach with appropriate a posterior error estimators for the approximate solution on the skeleton of the domain decomposition; and to estimate the finite and boundary element approximations of the local Steklov–Poincaré operators.

Finally, a challenging task is the implementation of the proposed algorithms for hybrid coupled domain decomposition methods in three space dimensions, including the coupling of finite and boundary element methods. For parallel implementations of domain decomposition methods based on finite elements we refer, e.g., to [11, 40]. The software package *ug* is a general finite element toolbox [5] which was used to implement also the Mortar finite element method in 2D, see [74]. The implementation of hybrid coupled domain decomposition methods in 3D is an actual reasearch topic of several groups. When these methods are available, they can provide a base for more complicated algorithms for the numerical solution of boundary value problems arising in, for example mechanical, applications.

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